Micro Qualifer Notes

CHAPTER 1. PREFERENCE AND CHOICE

Definition 1.B.1: The preference relation $\succsim$ is **rational** if it possesses the following two properties:
(i) Completeness: for all $x, y \in X$, we have that $x \succsim y$ or $y \succsim x$ (or both)
(ii) Transitivity: for all $x, y, z \in X$, if $x \succsim y$ and $y \succsim z$, then $x \succsim z$.

Proposition 1.B.1: If $\succsim$ is rational then:
(i) $\succ$ is both **irreflexive** ($x \succ x$ never holds) and **transitive** (if $x \succ y$ and $y \succ z$, then $x \succ z$).
(ii) $\sim$ is **reflexive** ($x \sim x$ for all x), **transitive**, and **symmetric** (if $x \sim y$ then $y \sim x$).
(iii) $x \succ y \succsim z$, then $x \succ z$.

Definition 1.B.2: A **utility function** $u : X \longrightarrow \mathbb{R}$ **represents** a preference relation $\succsim$ if $x \succsim y$ implies $u(x) \geq u(y)$ for all $x, y \in X$ and vice versa.

Proposition 1.B.2: A preference relation $\succsim$ can be represented by a utility function only if it is rational.

Definition 1.C.1: The choice structure $(\mathscr{B}, C(\cdot))$ satisfies the **weak axiom of revealed preference (WARP)** if $x, y \in B$ and $x \in C(B)$ for some $B \in \mathscr{B}$ implies $y \in C(B')$ for all $B' \in \mathscr{B}$ such that $x, y \in B'$.

Definition 1.C.2: Given a choice structure $(\mathscr{B}, C(\cdot))$, x is **revealed preferred** to y (*i.e.* $x \succsim^* y$) if there is some $B \in \mathscr{B}$ such that $x, y \in B$ and $x \in C(B)$.

Proposition 1.D.1: Suppose $\succsim$ is rational. Then the choice structure $(\mathscr{B}, C^*(\cdot, \succsim))$ generated by $\succsim$ satisfies the weak axiom.

Proposition 1.D.2: Suppose $(\mathscr{B}, C(\cdot))$ satisfies the weak axiom, and $\mathscr{B}$ includes all subsets of X of up to three elements. Then there is a rational preference relation $\succsim$ that rationalizes $C(\cdot)$ relative to $\mathscr{B}$, that is, $C(B) = C^*(B, \succsim)$ for all $B \in \mathscr{B}$. Furthermore, this $\succsim$ is the only such preference relation.

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Definition 2.D.1: The **Walrasian** or **competitive budget set** $B_{p,w} = \{x \in \mathbb{R}^L_+ : p \cdot x \leq w\}$ is the set of all feasible consumption bundles for the consumer who faces market prices p and has wealth w.

Definition 2.E.1: The Walrasian demand correspondence $x(p, w)$ is **homogeneous of degree zero** if $x(\alpha p, \alpha w) = x(p, w)$ for any p, w and $\alpha > 0$.

Definition 2.E.2: The Walrasian demand correspondence $x(p, w)$ satisfies **Walras' law** if for every $p \gg 0$ and $w > 0$, we have $p \cdot x = w$ for all $x \in x(p, w)$.

Definition: A commodity l is **normal** at (p, w) if $\partial x_l(p, w)/\partial w \geq 0$; that is, demand is nondecreasing in welath. If commodity l's wealth effect is instead negative, then it is called **inferior** at (p, w). If every commodity is normal at all (p, w), then we say that **demand is normal**. Good l is said to be a **Giffen good** at (p, w) if $\partial x_l(p, w)/\partial p_l > 0$.

Definition: A commodity l is a **necessary good** if

$$\varepsilon_{lw} = \frac{\partial x_l(p, w)}{\partial w} \frac{w}{x_l(p, w)} \leq 1,$$

and a **luxury good** if $\varepsilon_{lw} > 1$.

Proposition 2.E.1: If the Walrasian demand function $x(p, w)$ is homogeneous of degree zero, then for all p and w:

$$\sum_{k=1}^{L} \frac{\partial x_l(p, w)}{\partial p_k} p_k + \frac{\partial x_l(p, w)}{\partial w} w = 0 \text{ for } l = 1, \ldots, L,$$

or, in matrix notation,

$$D_p x(p, w) \cdot p + D_w x(p, w) w = 0.$$

Proposition 2.E.2: If the Walrasian demand function $x(p, w)$ satisfies Walras' law, then for all p and w:

$$\sum_{l=1}^{L} p_l \frac{\partial x_l(p, w)}{\partial p_k} + x_k(p, w) = 0 \text{ for } k = 1, \ldots, L\cdot$$

or, in matrix notation,

$$p \cdot D_p x(p, w) + x(p, w)^{\mathrm{T}} = 0^{\mathrm{T}}.$$

Proposition 2.E.3: If the Walrasian demand function $x(p, w)$ satisfies Walras' law, then for all p and w:

$$\sum_{l=1}^{L} p_l \frac{\partial x_l(p,w)}{\partial w} = 1$$

or, in matrix notation,

$$p \cdot D_w x(p,w)^{\mathrm{T}} = 1.$$

Definition 2.F.1: The Walrasian demand function $x(p,w)$ satisfies the **weak axiom of revealed preference (WARP)** if $p \cdot x(p',w') \le w$ and $x(p',w') \ne x(p,w)$ imply $p' \cdot x(p,w) > w'$.

Proposition 2.F.1: Suppose the Walrasian demand function $x(p,w)$ is homogeneous of degree zero and satisfies Walras' law. Let (p,w) and $(p',w') = (p', p' \cdot x(p,w))$ be a compensated price-wealth change. Then $x(p,w)$ satisfies the weak axiom if and only if

$$(p' - p) \cdot [x(p',w') - x(p,w)] \le 0$$

with strict inequality whenever $x(p,w) \ne x(p',w')$.

Proposition 2.F.2: If a differentiable Walrasian demand function $x(p,w)$ satisfies Walras' law, homogeneity of degree zero, and the weak axiom, then at any (p,w), the Slutsky matrix $S(p,w)$ satisfies $v \cdot S(p,w)v \le 0$ for any $v \in \mathbb{R}^L$.

Proposition 2.F.3: Suppose that the Walrasian demand function $x(p,w)$ is differentiable, homogeneous of degree zero, and satisfies Walras' law. Then $p \cdot S(p,w) = 0$ and $S(p,w)p = 0$ for any (p,w).

CHAPTER 3. CLASSICAL DEMAND THEORY

Definition 3.B.1: The preference relation $\succsim$ is **rational** if it possesses the following two properties:
(i) Completeness: for all $x, y \in X$, we have that $x \succsim y$ or $y \succsim x$ (or both)
(ii) Transitivity: for all $x, y, z \in X$, if $x \succsim y$ and $y \succsim z$, then $x \succsim z$.

Definition 3.B.2: The preference relation $\succsim$ on X is **monotone** if $x \in X$ and $y \gg x$ implies $y \succ x$. It is **strongly monotone** if $y \ge x$ and $y \ne x$ imply that $y \succ x$.

Definition 3.B.3: The preference relation $\succsim$ on X is **locally nonsatiated** if for every $x \in X$ and every $\epsilon > 0$, there is $y \in X$ such that $\|y - x\| \le \epsilon$ and $y \succ x$.

Definition 3.B.4: The preference relation $\succsim$ on X is **convex** if for every $x \in X$, the upper contour set $\{y \in X : y \succsim x\}$ is convex; that is, if $y \succsim x$ and $z \succsim x$, then $\alpha y + (1-\alpha)z \succsim x$ for any $\alpha \in [0,1]$.

Definition 3.B.5: The preference relation $\succsim$ on X is **strictly convex** if for every x, we have that $y \succsim x$, $z \succsim x$, and $y \neq z$ implies $\alpha y + (1-\alpha)z \succ x$ for all $\alpha \in (0,1)$.

Definition 3.B.6: A monotone preference relation $\succsim$ on X is **homothetic** if all indifference sets are related by proportional expansion along rays; that is, if $x \sim y$, then $\alpha x \sim \alpha y$ for any $\alpha \geq 0$.

Definition 3.B.7: The preference relation $\succsim$ on $X = (-\infty, \infty) \times \mathbb{R}_+^{L-1}$ is **quasilinear** with respect to commodity 1 (called, in this case, the **numeraire** commodity) if
(i) All the indifference sets are parallel displacements of each other along the axis of commodity 1. That is, if $x \sim y$, then $(x + \alpha e_1) \sim (y + \alpha e_1)$ for $e_1 = (1, 0, \ldots, 0)$ and any $\alpha \in \mathbb{R}$.
(ii) Good 1 is **desirable**, that is, $x + \alpha e_1 \succ x$ for all x and $\alpha > 0$.

Definition 3.C.1: The preference relation $\succsim$ on X is **continuous** if it is preserved under limits. That is, for any sequence of pairs $\{(x^n, y^n)\}_{n=1}^{\infty}$ with $x^n \succsim y^n$ for all n, $x = \lim_{n\to\infty} x^n$, and $y = \lim_{n\to\infty} y^n$. we have $x \succsim y$.

Proposition 3.C.1: Suppose that the rational preference relation $\succsim$ on X is continuous. Then there is a continuous utility function $u(x)$ that represents $\succsim$.

Proposition: **(i)** A continuous $\succsim$ on $X = \mathbb{R}_+^L$ is homothetic if and only if it admits a utility function $u(x)$ that is homogeneous of degree one. **(ii)** A continuous $\succsim$ on $(-\infty, \infty) \times \mathbb{R}_+^{L-1}$ is quasilinear with respect to the first commodity if and only if it admits a utility function $u(x)$ of the form $u(x) = x_1 + \phi(x_2, \ldots, x_L)$.

Proposition 3.D.1: If $p \gg 0$ and $u(\cdot)$ is continuous, then the utility maximization problem has a solution.

Proposition 3.D.2: Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}_+^L$. Then the Walrasian demand correspondence $x(p, w)$ possesses the following properties:
(i) Homogeneity of degree zero in (p, w): $x(\alpha p, \alpha w) = x(p, w)$ for any p, w and scalar $\alpha > 0$.
(ii) Convexity/uniqueness: If $\succsim$ is convex, so that $u(\cdot)$ is quasiconcave, then $x(p, w)$ is a convex set. Moreover, if $\succsim$ is strictly convex, so that $u(\cdot)$ is strictly quasiconcave, then $x(p, w)$ consists of a single element.

Proposition 3.D.3: Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}_+^L$. The indirect utility function $v(p, w)$ is **(i)** Homogeneous of degree zero. **(ii)** Strictly increasing in w and nonincreasing in p_l for any l. **(iii)** Quasiconvex; that is, the set $\{(p, w) : v(p, w) \leq \bar{v}\}$ is convex for any $\bar{v}$. **(iv)** Continuous in p and w.

Proposition 3.E.1: Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$ and that the price vector is $p \gg 0$. We have
(i) If x^* is optimal in the UMP when wealth is $w > 0$, then x^* is optimal in the EMP when the required utility level is $u(x^*)$. Moreover, the minimized expenditure level in this EMP is exactly w.
(ii) If x^* is optimal in the EMP when the required utility level is $u > u(0)$, then x^* is optimal in the UMP when wealth is $p \cdot x^*$. Moreover, the maximized utility level in this UMP is exactly u.

Proposition 3.E.2: Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$. The expenditure function $e(p, u)$ is **(i)** Homogeneous of degree one in p. **(ii)** Strictly increasing in u and nondecreasing in p_l for any l. **(iii)** Concave in p. **(iv)** Continuous in p and u.

Proposition 3.E.3: Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$. Then for any $p \gg 0$, the Hicksian demand correspondence $h(p, u)$ possesses the following properties: **(i)** Homogeneity of degree zero in p: $h(\alpha p, u) = h(p, u)$ for any p, u and $\alpha > 0$. **(ii)** No excess utility: for any $x \in h(p, u)$, $u(x) = u$. **(iii)** Convexity/uniqueness: if $\succsim$ is convex, then $h(p, u)$ is a convex set; and if $\succsim$ is strictly convex, so that $u(\cdot)$ is strictly quasiconcave, then there is a unique element in $h(p, u)$.

Proposition 3.E.4: Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ and that $h(p, u)$ consists of a single element for all $p \gg 0$. Then the Hicksian demand function $h(p, u)$ satisfies the compensated law of demand: for all p' and p'',

$$(p'' - p') \cdot [h(p'', u) - h(p', u)] \leq 0.$$

Definition 3.F.1: For any nonempty closed set K, the **support function** of K is defined for any $p \in \mathbb{R}^L$ to be $\mu_K(p) = \inf\{p \cdot x : x \in K\}$.

Proposition 3.F.1: (**The Duality Theorem**) Let K be a nonempty closed set, and let $\mu_K(\cdot)$ be its support function. Then there is a unique $\bar{x} \in K$ such that $\bar{p} \cdot \bar{x} = \mu_K(\bar{p})$ if and only if $\mu_K(\cdot)$ is differentiable at $\bar{p}$. Moreover, in this case, $\nabla \mu_K(\bar{p}) = \bar{x}$.

Proposition 3.G.1: Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$. For all p and u, the Hicksian demand $h(p,u)$ is the derivative vector of the expenditure function with respect to prices:

$$h(p,u) = \nabla_p e(p,u).$$

That is, $h_l(p,u) = \partial e(p,u)/\partial p_l$ for all $l = 1,\ldots,L$.

Proposition 3.G.2: Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$. Suppose also that $h(\cdot,u)$ is continuously differentiable at (p,u), and denote its $L \times L$ derivative matrix by $D_p h(p,u)$. Then **(i)** $D_p h(p,u) = D_p^2 e(p,u)$. **(ii)** $D_p h(p,u)$ is a negative semidefinite matrix. **(iii)** $D_p h(p,u)$ is a symmetric matrix. **(iv)** $D_p h(p,u)p = 0$.

Proposition 3.G.3: (The Slutsky Equation) Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated and strictly convex preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$. Then for all (p,w), and $u = v(p,w)$, we have

$$\frac{\partial h_l(p,u)}{\partial p_k} = \frac{\partial x_l(p,w)}{\partial p_k} + \frac{\partial x_l(p,w)}{\partial w} x_k(p,w) \quad \text{for all } l,\ k$$

or equivalently, in matrix notation,

$$D_p h(p,u) = D_p x(p,w) + D_w x(p,w) x(p,w)^{\mathrm{T}}.$$

Proposition 3.G.4: (Roy's Identity) Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated and strictly convex preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$. Suppose also that the indirect utility function is differentiable at $(\bar{p},\bar{w}) \gg 0$. Then

$$x(\bar{p},\bar{w}) = -\frac{1}{\nabla_w v(\bar{p},\bar{w})} \nabla_p v(\bar{p},\bar{w}).$$

That is, for every $l = 1,\ldots,L$:

$$x_l(\bar{p},\bar{w}) = -\frac{\partial v(\bar{p},\bar{w})/\partial p_l}{\partial v(\bar{p},\bar{w})/\partial w}.$$

Proposition 3.H.1: Suppose that $e(p,u)$ is strictly increasing in u and is continuous, increasing, homogeneous of degree one, concave, and differentiable in p. Then, for every utility level u, $e(p,u)$ is the expenditure function associated with the at-least-as-good-as set

$$V_u = \{x \in \mathbb{R}^L_+ : p \cdot x \geq e(p,u) \text{ for all } p \gg 0\}.$$

That is, $e(p,u) = \min\{p \cdot x : x \in V_u\}$ for all $p \gg 0$.

Proposition 3.I.1: Suppose that the consumer has a locally nonsatiated rational preference relation $\succsim$. If $(p^1 - p^0) \cdot x^0 < 0$, then the consumer is strictly better off under price-wealth situation (p^1, w) than under (p^0, w).

Proposition 3.I.2: Suppose that the consumer has a differentiable expenditure function. Then if $(p^1 - p^0) \cdot x^0 > 0$, there is a sufficiently small $\bar{\alpha} \in (0, 1)$ such that for all $\alpha < \bar{\alpha}$, we have $e((1-\alpha)p^0 + \alpha p^1, u^0) > w$

Definition: An **equivalent variation (EV)** is the dollar amount that the consumer would be indifferent about accepting instead of the price change. A **compensating variation (CV)** is the negative of the amount that the consumer would be just willing to accept to allow the price change to happen. Formally, let p and p' denote the prices before and after the change. Let w be the initial wealth. Then EV and CV satisfy

$$v(p', w) = v(p, w + EV),$$
$$v(p, w) = v(p', w - CV).$$

Proposition: Let $u^i = v(p^i, w)$ and $e(p^i, u^i) = w$ for $i = 0, 1$. Then

$$EV(p^0, p^1, w) = e(p^0, u^1) - e(p^0, u^0) = e(p^0, u^1) - w,$$
$$CV(p^0, p^1, w) = e(p^1, u^1) - e(p^1, u^0) = w - e(p^1, u^0).$$

Definition 3.J.1: The market demand function $x(p, w)$ satisfies the **strong axiom of revealed preference (the SA)** if for any list

$$(p^1, w^1), \ldots, (p^N, w^N)$$

with $x(p^{n+1}, w^{n+1}) \neq x(p^n, w^n)$ for all $n \leq N - 1$, we have $p^N \cdot x(p^1, w^1) > w^N$ whenever $p^n \cdot x(p^{n+1}, w^{n+1}) \leq w^n$ for all $n \leq N - 1$.

Proposition 3.J.1: If the Walrasian demand function $x(p, w)$ satisfies the strong axiom of revealed preference then there is a rational preference relation $\succsim$ that rationalizes $x(p, w)$, that is, such that for all (p, w), $x(p, w) \succ y$ for every $y \neq x(p, w)$ with $y \in B_{p,w}$.

Definition 3.AA.1: The Walrasian demand correspondence $x(p, w)$ is **upper hemicontinuous** at $(\bar{p}, \bar{w})$ if whenever $(p^n, w^n) \longrightarrow (\bar{p}, \bar{w})$, $x^n \in x(p^n, w^n)$ for all n, and $x = \lim_{n \to \infty} x^n$, we have $x \in x(\bar{p}, \bar{w})$.

Proposition 3.AA.1: Suppose that $u(\cdot)$ is a continuous utility function representing locally nonsatiated preferences $\succsim$ on the consumption set $X = \mathbb{R}^L_+$. Then the derived demand correspondence $x(p, w)$ is upper hemicontinuous at all $(p, w) \gg 0$. Moreover, if $x(p, w)$ is a function, then it is continuous at all $(p, w) \gg 0$.

Proposition 4.B.1: A necessary and sufficient condition for the set of consumers to exhibit parallel, straight wealth expansion paths at any price vector p is that preferences admit indirect utility functions of the **Gorman form** with the coefficients on w_i the same for every consumer i. That is:

$$v_i(p, w_i) = a_1(p) + b(p)w_i$$

Definition 4.C.1: The aggregate demand function $x(p, w)$ satisfies the **weak axiom (WA)** if $p \cdot x(p', w') \leq w$ and $x(p, w) \neq x(p', w')$ imply $p' \cdot x(p, w) > w'$ for any (p, w) and (p', w').

Definition 4.C.2: The individual demand function $x_i(p, w_i)$ satisfies the **uncompensated law of demand (ULD)** property if

$$(p' - p) \cdot [x_i(p', w_i) - x_i(p, w_i)] \leq 0$$

for any p, p', and w_i with strict inequality if $x_i(p', w_i) \neq x_i(p, w_i)$. The analogous definition applies to the aggregate demand function $x(p, w)$.

Proposition 4.C.1: If every consumer's Walrasian demand function $x_i(p, w_i)$ satisfies the uncompensated law of demand (ULD) property, so does the aggregate demand $x(p, w) = \sum_i x_i(p, \alpha_i w)$. As a consequence, the aggregate demand $x(p, w)$ satisfies the weak axiom.

Proposition 4.C.2: If $\succsim_i$ is homothetic, then $x_i(p, w_i)$ satisfies the uncompensated law of demand (ULD) property.

Proposition 4.C.3: Suppose that $\succsim_i$ is defined on the consumption set $X = \mathbb{R}^L_+$ and is representable by a twice continuously differentiable concave function $u_j(\cdot)$. If

$$-\frac{x_i \cdot D^2 u_i(x_i) x_i}{x_i \cdot \nabla u_i(x_i)} < 4 \quad \text{for all } x_i,$$

then $x_i(p, w_i)$ satisfies the unrestricted law of demand (ULD) property.

Proposition 4.C.4: Suppose that all consumers have identical preferences $\succsim$ defined on $\mathbb{R}^L_+$ [with individual demand functions denoted $\tilde{x}(p, w)$] and that individual wealth is uniformly distributed on an interval $[0, \bar{w}]$ (strictly speaking, this requires a continuum of consumers). Then the aggregate (rigorously, the average) demand function

$$x(p) = \int_0^{\bar{w}} \tilde{x}(p, w) dw$$

satisfies the unrestricted law of demand (ULD) property.

Definition 4.D.1: A **positive representative consumer** exists if there is a rational preference relation $\succsim$ on $\mathbb{R}^L_+$ such that the aggregate demand function $x(p,w)$ is precisely the Walrasian demand function generated by this preference relation. That is, $x(p,w) \succ x$ whenever $x \neq x(p,w)$ and $p \cdot x \leq w$.

Definition 4.D.2: A **(Bergson-Samuelson) social welfare function** is a function $W : \mathbb{R}^I \longrightarrow \mathbb{R}$ that assigns a utility value to each possible vector $(u_1, \ldots, u_I) \in \mathbb{R}^I$ of utility levels for the I consumers in the economy.

Proposition 4.D.1: Suppose that for each level of prices p and aggregate wealth w, the wealth distribution $(w_1(p,w), \ldots, w_I(p,w))$ solves the problem

$$\max_{w_1,\ldots,w_I} W\Big(v_1(p,w_1), \cdots, v_I(p,w_I)\Big)$$

subject to $\sum_{i=1}^I w_i \leq w$, where $v_i(p,w)$ is consumer i's indirect utility function. Then the value function $v(p,w)$ is an indirect utility function of a positive representative consumer for the aggregate demand function $x(p,w) = \sum_i x_i\big(p, w_i(p,w)\big)$.

Definition 4.D.3: The positive representative consumer $\succsim$ for the aggregate demand $x(p,w) = \sum_i x_i\big(p, w_i(p,w)\big)$ is a **normative representative consumer** relative to the social welfare function $W(\cdot)$ if for every (p,w), the distribution of wealth $\big(w_1(p,w), \ldots, w_I(p,w)\big)$ solves the problem

$$\max_{w_1,\ldots,w_I} W\Big(v_1(p,w_1), \cdots, v_I(p,w_I)\Big)$$

subject to $\sum_{i=1}^I w_i \leq w$, where $v_i(p,w)$ is consumer i's indirect utility function, and, therefore, the value function of the above problem is an indirect utility function for $\succsim$.

CHAPTER 5. PRODUCTION

Definition: The **properties of production sets** are: (i) Y is nonempty, (ii) Y is closed, (iii) no free lunch, (iv) possibility of inaction, (v) free disposal, (vi) irreversibility, (vii) nonincreasing returns to scale, (viii) nondecreasing returns to scale, (xi) constant returns to scale (Y is a cone), (x) additivity (or free entry), (xi) convexity, and (xii) Y is a convex cone.

Proposition 5.B.1: The production set Y is additive and satisfies the nonincreasing returns condition if and only if it is a convex cone.

Proposition 5.B.2: For any convex production set $Y \subset \mathbb{R}^L$ with $0 \in Y$, there is a constant returns, convex production set $Y' \subset \mathbb{R}^{L+1}$ such that $Y = \{y \in \mathbb{R}^L : (y, -1) \in Y'\}$.

Definition: Given technological constraints represented by its production set Y, the firm's **profit maximization problem (PMP)** is

$$\max_{y} \; p \cdot y \qquad \text{s.t.} \qquad y \in Y.$$

Using a transformation function $F(\cdot)$ to describe Y, we can equivalently state the PMP as

$$\max_{y} \; p \cdot y \qquad \text{s.t.} \qquad F(y) \leq 0.$$

In the above problem, the resulting maximum $\pi(p)$ is called the firm's **profit function**; the maximizer $y(p)$ is called the **supply correspondence**. When Y corresponds to a single-output technology with differentiable production function $f(z)$, the input vector z^* maximizes profit given (p, w) if it solves

$$\max_{z \geq 0} \; pf(z) - w \cdot z.$$

Proposition 5.C.1: Suppose that $\pi(\cdot)$ is the profit function of the production set Y and that $y(\cdot)$ is the associated supply correspondence. Assume also that Y is closed and satisfies the free disposal property. Then: **(i)** $\pi(\cdot)$ is homogeneous of degree one. **(ii)** $\pi(\cdot)$ is convex. **(iii)** If Y is convex, then $Y = \{y \in \mathbb{R}^L : p \cdot y \leq \pi(p) \text{ for all } p \gg 0\}$. **(iv)** $y(\cdot)$ is homogeneous of degree zero. **(v)** If Y is convex, then $y(p)$ is a convex set for all p. Moreover, if Y is strictly convex, then $y(p)$ is single-valued (if nonempty). **(vi)** (**Hotelling's lemma**) If $y(\bar{p})$ consists of a single point, then $\pi(\cdot)$ is differentiable at $\bar{p}$ and $\nabla\pi(\bar{p}) = y(\bar{p})$. **(vii)** If $y(\cdot)$ is a function differentiable at $\bar{p}$, then $Dy(\bar{p}) = D^2\pi(\bar{p})$ is a symmetric and positive semidefinite matrix with $Dy(\bar{p}) \cdot \bar{p} = 0$.

Definition: The **cost minimization problem (CMP)** can be stated as follows (assume free disposal of output):

$$\min_{z \geq 0} \; w \cdot z \qquad \text{s.t.} \qquad f(z) \geq q.$$

The optimized value of the CMP is given by the **cost function** $c(w, q)$. The corresponding optimizing set of input (or factor) choices, denoted by $z(w, q)$ is known as the **conditional factor demand correspondence**.

Proposition 5.C.2: Suppose that $c(w, q)$ is the cost function of a single-output technology Y with production function $f(\cdot)$ and that $z(w, q)$ is the associated conditional factor demand correspondence. Assume also that Y is closed and satisfies the free disposal property. Then:

(i) $c(\cdot)$ is homogeneous of degree one in w and nondecreasing in q.

(ii) $c(\cdot)$ is a concave function of w.

(iii) If the sets $\{z \geq 0 : f(z) \geq q\}$ are convex for every q, then

$$Y = \{(-z, q) : w \cdot z \geq c(w, q) \text{ for all } \gg\}.$$

(iv) $z(\cdot)$ is homogeneous of degree zero in w.

(v) If the set $\{z \geq 0 : f(z) \geq q\}$ is convex, then $z(w, q)$ is a convex set. Moreover, if $\{z \geq 0 : f(z) \geq q\}$ is a strictly convex set, then $z(w, q)$ is single-valued.
(vi) (**Shephard's lemma**) If $z(\overline{w}, q)$ consists of a single point, then $c(\cdot)$ is differentiable with respect to w at $\overline{w}$ and $\nabla_w c(\overline{w}, q) = z(\overline{w}, q)$.
(vii) If $z(\cdot)$ is differentiable at $\overline{w}$, then $D_w z(\overline{w}, q) = D_w^2 c(\overline{w}, q)$ is a symmetric and negative semidefinite matrix with $D_w z(\overline{w}, q) \cdot \overline{w} = 0$.
(viii) If $f(\cdot)$ is homogeneous of degree one (i.e., exhibits constant returns to scale), then $c(\cdot)$ and $z(\cdot)$ are homogeneous of degree one in q.
(ix) If $f(\cdot)$ is concave, then $c(\cdot)$ is a convex function of q (in particular, marginal costs are nondecreasing in q).

CHAPTER 6. CHOICE UNDER UNCERTAINTY

Definition 6.B.1: A **simple lottery** $\boldsymbol{L}$ is a list $L = (p_1, ..., p_N)$ with $p_n \geq 0$ for all n and $\sum_n p_n = 1$, where p_n is interpreted as the probability of outcome n occurring.

Definition 6.B.2: Given K simple lotteries $L_k = (p_1^k, \ldots, p_N^k)$, $k = 1, \ldots, K$, and probabilities $\alpha_k \geq 0$ with $\sum_k \alpha_k = 1$, the **compound lottery** $(L_1, \ldots, L_k; \alpha_1, \ldots, \alpha_K)$ is the risky alternative that yields the simple lottery L_k with probability α_k for $k = 1, \ldots, K$.

Definition 6.B.3: The preference relation $\succsim$ on the space of simple lotteries $\mathscr{L}$ is **continuous** if for any $L, L', L'' \in \mathscr{L}$, the sets

$$\{\alpha \in [0,1] : \alpha L + (1-\alpha)L' \succsim L''\} \subset [0,1],$$

and

$$\{\alpha \in [0,1] : L'' \succsim \alpha L + (1-\alpha)L'\} \subset [0,1]$$

are closed.

Definition 6.B.4: The preference relation $\succsim$ on the space of simple lotteries $\mathscr{L}$ satisfies the **independence axiom** if for all $L, L', L'' \in \mathscr{L}$ and $\alpha \in (0,1)$ we have

$$L \succsim L' \quad \text{if and only if } \alpha L + (1-\alpha)L'' \succsim \alpha L' + (1-\alpha)L''.$$

Definition 6.B.5: The utility function $U : \mathscr{L} \longrightarrow \mathbb{R}$ has an **expected utility form** if there is an assignment of numbers $(u_1, \ldots, u_N)$ to the N outcomes such that for every simple lottery $L = (p_1, \ldots, p_N) \in \mathscr{L}$ we have

$$U(L) = u_1 p_1 + \cdots + u_N p_N.$$

A utility function $U : \mathscr{L} \longrightarrow \mathbb{R}$ with the expected utility form is called a **von Neumann-Morgenstern (v.N-M) expected utility function**.

Proposition 6.B.1: A utility function $U : \mathscr{L} \longrightarrow \mathbb{R}$ has an expected utility form if and only if it is linear, that is, if and only if it satisfies the property that

$$U\left(\sum_{k=1}^{K} \alpha_K L_k\right) = \sum_{k=1}^{K} \alpha_k U(L_k)$$

for any K lotteries $L_k \in \mathscr{L}$, $k = 1, \cdots, K$, and probabilities $(\alpha_1, \cdots, \alpha_K) \geq 0$, $\sum_k \alpha_k = 1$.

Proposition 6.B.2: A utility function $U : \mathscr{L} \longrightarrow \mathbb{R}$ is a v.N-M expected utility function for the preference relation $\succsim$ on $\mathscr{L}$. Then $\widetilde{U} : \mathscr{L} \longrightarrow \mathbb{R}$ is another v.N-M utility function for $\succsim$ if and only if there are scalars $\beta > 0$ and γ such that $\widetilde{U}(L) = \beta U(L) + \gamma$ for every $L \in \mathscr{L}$.

Proposition 6.B.3: (Expected Utility Theorem) Suppose that the rational preference relation $\succsim$ on the space of lotteries $\mathscr{L}$ satisfies the continuity and independence axioms. Then $\succsim$ admits a utility representation of the expected utility form. That is, we can assign a number u_n to each outcome $n = 1, \ldots, N$ in such a manner that for any two lotteries $L = (p_1, \ldots, p_N)$ and $L' = (p'_1, \ldots, p'_N)$, we have

$$L \succsim L' \quad \text{if and only if} \quad \sum_{n=1}^{N} u_n p_n \geq \sum_{n=1}^{N} u_n p'_n.$$

Definition 6.C.1: A decision maker is a **risk averter** (or exhibits **risk aversion**) if for any lottery $F(\cdot)$, the degenerate lottery that yields the amount $\int x dF(x)$ with certainty is at least as good as the lottery $F(\cdot)$ itself. If the decision maker is always (*i.e.*, for any $F(\cdot)$) indifferent between these two lotteries, we say that he is **risk neutral**. Finally, we say that he is **strictly risk averse** if indifference holds only when the two lotteries are the same (*i.e.*, when $F(\cdot)$ is degenerate).

Definition 6.C.2: Given a Bernoulli utility function $u(\cdot)$ we define the following concepts:
(i) The **certainty equivalent** of $F(\cdot)$, denoted $c(F, u)$, is the amount of money for which the individual is indifferent between the gamble $F(\cdot)$ and the certain amount $c(F, u)$; that is,

$$u(c(F, u)) = \int u(x) dF(x).$$

(ii) For any fixed amount of money x and positive number ϵ, the **probability premium** denoted by $\pi(x, \epsilon, u)$, is the excess in winning probability over fair odds that makes the individual indifferent between the certain outcome x and a gamble between the two outcomes $x + \epsilon$ and $x - \epsilon$. That is

$$u(x) = \left(\frac{1}{2} + \pi(x, \epsilon, u)\right) u(x + \epsilon) + \left(\frac{1}{2} - \pi(x, \epsilon, u)\right) u(x - \epsilon).$$

Proposition 6.C.1: Suppose a decision maker is an expected utility maximizer with a Bernoulli utility function $u(\cdot)$ on amounts of money. Then the following properties are equivalent:
(i) The decision maker is risk averse.

(ii) $u(\cdot)$ is concave.
(iii) $c(F, u) \leq \int x dF(x)$ for all $F(\cdot)$.
(iv) $\pi(x, \epsilon, u) \geq 0$ for all x, ϵ.

Definition 6.C.3: Given a (twice-differentiable) Bernoulli utility function $u(\cdot)$ for money, the **Arror-Pratt coefficient of absolute risk aversion** at x is defined as

$$r_A(x) = \frac{-u''(x)}{u'(x)}.$$

Proposition 6.C.2: The following definitions of the more-risk-averse-than relation are equivalent.
(i) $r_A(x, u_2) \geq r_A(x, u_1)$ for every x.
(ii) There exists an increasing concave function $\psi(\cdot)$ such that $u_2(x) = \psi(u_1(x))$ at all x; that is, $u_2(\cdot)$ is a concave transformation of $u_1(\cdot)$. [In other words, $u_2(\cdot)$ is "more concave" than $u_1(\cdot)$.
(iii) $c(F, u_2) \leq c(F, u_1)$ for any $F(\cdot)$.
(iv) $\pi(x, \varepsilon, u_2) \geq \pi(x, \varepsilon, u_1)$ for any x and ε.
(v) Whenever $u_2(\cdot)$ finds a lottery $F(\cdot)$ at least as good as a riskless outcome $\bar{x}$, then $u_1(\cdot)$ also finds $F(\cdot)$ at least as good as $\bar{x}$. That is, $\int u_2(x) dF(x) \geq u_2(\bar{x})$ implies $\int u_1(x) dF(x) \geq u_1(\bar{x})$ for any $F(\cdot)$ and $\bar{x}$.

Definition 6.C.4: The Bernoulli utility function $u(\cdot)$ for money exhibits **decreasing absolute risk aversion** if $r_A(x, u)$ is a decreasing function of x.

Proposition 6.C.3: The following properties are equivalent:
(i) The Bernoulli utility function $u(\cdot)$ exhibits decreasing absolute risk aversion.
(ii) Whenever $x_2 < x_1$, $u_2(z) = u(x_2 + z)$ is a concave transformation of $u_1(z) = u(x_1 + z)$.
(iii) For any risk $F(z)$, the certainty equivalent of the lottery formed by adding risk z to wealth level x, given by the amount c_x at which $u(c_x) = \int u(x + z) dF(z)$, is such that $(x - c_x)$ is decreasing in x. That is, the higher x is, the less is the individual willing to pay to get rid of the risk.
(iv) The probability premium $\pi(x, \varepsilon, u)$ is decreasing in x.
(v) For any $F(z)$, if $\int u(x_2 + z) dF(z) \geq u(x_2)$ and $x_2 < x_1$, then $\int u(x_1 + z) dF(z) \geq u(x_1)$.

Definition 6.C.5: Given a Bernoulli utility function $u(\cdot)$, the **coefficient of relative risk aversion** at x is

$$r_R(x, u) = -\frac{x u''(x)}{u'(x)}.$$

Proposition 6.C.4: The following conditions for a Bernoulli utility function $u(\cdot)$ on amounts of money are equivalent:
(i) $r_R(x,u)$ is decreasing in x.
(ii) Whenever $x_2 < x_1$, $\tilde{u}_2(t) = u(tx_2)$ is a concave transformation of $\tilde{u}_1(t) = u(tx_1)$.
(iii) Given any risk $F(t)$ on $t > 0$, the certainty equivalent $\bar{c}_x$ defined by

$$u(\bar{c}_x) = \int u(tx)dF(t)$$

is such that $x/\bar{c}_x$ is decreasing in x.

Definition 6.D.1: The distribution $F(\cdot)$ **first-order stochastically dominates** $G(\cdot)$ if, for every nondecreasing function $u : \mathbb{R} \longrightarrow \mathbb{R}$, we have

$$\int u(x)dF(x) \geq \int u(x)dG(x).$$

Proposition 6.D.1: The distribution of monetary payoffs $F(\cdot)$ first-order stochastically dominates the distribution $G(\cdot)$ if and only if $F(x) \leq G(x)$ for every x.

Definition 6.D.2: For any two distributions $F(x)$ and $G(x)$ with the same mean, $F(\cdot)$ **second-order stochastically dominates** (or **is less risky than**) $G(\cdot)$ if for every nondecreasing concave function $u : \mathbb{R}_+ \longrightarrow \mathbb{R}$, we have

$$\int u(x)dF(x) \geq \int u(x)dG(x).$$

Proposition 6.D.2: Consider two distributions $F(x)$ and $G(x)$ with the same mean. Then the following statements are equivalent:
(i) $F(\cdot)$ second-order stochastically dominates $G(\cdot)$.
(ii) $G(\cdot)$ is a mean-preserving spread of $F(\cdot)$.
(iii) For all x,

$$\int_0^x G(t)dt \geq \int_0^x F(t)dt.$$

Definition 6.E.1: A **random variable** is a function $g : S \longrightarrow \mathbb{R}_+$ that maps states into monetary outcomes.

Definition 6.E.2: The preference relation $\succsim$ has an **extended expected utility representation** if for every $s \in S$, there is a function $u_s : \mathbb{R}_+ \longrightarrow \mathbb{R}$ such that for any $(x_1, \ldots, x_S) \in \mathbb{R}_+^S$ and $(x'_1, \ldots, x'_S) \in \mathbb{R}_+^S$,

$$(x_1, \ldots, x_s) \succ (x'_1, \ldots, x'_S) \quad \text{if and only if} \quad \sum_s \pi_s u_s(x_s) \geq \sum_s \pi_s u_s(x'_s).$$

Definition 6.E.3: The preference relation $\succsim$ on $\mathscr{L}$ satisfies the **extended independence axiom** if for all $L, L', L'' \in \mathscr{L}$ and $\alpha \in (0,1)$, we have

$$L \succsim L' \quad \text{if and only if} \quad \alpha L + (1-\alpha)L'' \succsim \alpha L' + (1-\alpha)L''.$$

Proposition 6.E.1: Suppose that the preference relation $\succsim$ on $\mathscr{L}$ satisfies the continuity and extended independence axioms. Then we can assign a utility function $u_s(\cdot)$ for money in every state s such that for any $L = (F_1, \ldots, F_S)$ and $L' = (F'_1, \ldots, F'_S)$, we have

$$L \succsim L' \quad \text{if and only if} \quad \sum_s \left(\int u_s(x_s) dF_s(x_s) \right) \geq \sum_s \left(\int u_s(x_s) dF'_s(x_s) \right).$$

Definition 6.E.4: The preference relation $\succsim$ satisfies the **sure-thing axiom** if, for any subset of states $E \subset S$ (E is called an **event**), whenever $(x_1, \ldots, x_S)$ and $(x'_1, \ldots, x'_S)$ differ only in the entries corresponding to E (so that $x'_s = x_s$ for $s \notin E$), the preference ordering between $(x_1, \ldots, x_S)$ and $(x'_1, \ldots, x'_S)$ is independent of the particular (common) payoffs for states not in E. Formally, suppose that $(x_1, \ldots, x_S)$, $(x'_1, \ldots, x'_S)$, $(\bar{x}_1, \ldots, \bar{x}_S)$, and $(\bar{x}'_1, \ldots, \bar{x}'_S)$ are such that

For all $s \notin E$: $x_s = x'_s$ and $\bar{x}_s = \bar{x}'_s$.

For all $s \in E$: $x_s = \bar{x}_s$ and $x'_s = \bar{x}'_s$.

Then $(\bar{x}_1, \ldots, \bar{x}_S) \succsim (\bar{x}'_1, \ldots, \bar{x}'_S)$ if and only if $(x_1, \ldots, x_S) \succsim (x'_1, \ldots, x'_S)$.

Proposition 6.E.2: Suppose that there are at least three states and that the preferences $\succsim$ on $\mathbb{R}^S_+$ are continuous and satisfy the sure-thing axiom. Then $\succsim$ admits an extended expected utility representation.

Definition 6.F.1: The state preferences $(\succsim_1, \ldots, \succsim_S)$ on state lotteries are **state uniform** if $\succsim_s = \succsim_{s'}$ for any s and s'.

Proposition 6.F.1: (Subjective Expected Utility Theorem) Suppose that the preference relation $\succsim$ on $\mathscr{L}$ satisfies the continuity and extended independence axioms. Suppose, in addition, that the derived state preferences are state uniform. Then there are probabilities $(\pi_1, \ldots, \pi_S) \gg 0$ and a utility function $u(\cdot)$ on amounts of money such that for any $(x_1, \ldots, x_S)$ and $(x'_1, \ldots, x'_S)$ we have

$$(x_1, \ldots, x_S) \succsim (x'_1, \ldots, x'_S) \quad \text{if and only if} \quad \textstyle\sum_s \pi_s u(x_s) \geq \sum_s \pi_s u(x'_s).$$

Definition 8.B.1: A strategy $s_i \in S_i$ is a **strictly dominant strategy** for player i in game $\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]$ if for all $s_i' \neq s_i$, we have

$$u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$$

for all $s_{-i} \in S_{-i}$.

Definition 8.B.2: A strategy $s_i \in S_i$ is a **strictly dominated** for player i in game $\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]$ if there exists another strategy $s_i' \in S_i$ such that for all $s_{-i} \in S_{-i}$,

$$u_i(s_i', s_{-i}) > u_i(s_i, s_{-i}).$$

In this case, we say that strategy s_i' **strictly dominates** strategy s_i.

Definition 8.B.3: A strategy $s_i \in S_i$ is **weakly dominated** in game $\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]$ if there exists another strategy $s_i' \in S_i$ such that for all $s_{-i} \in S_{-i}$,

$$u_i(s_i', s_{-i}) \geq u_i(s_i, s_{-i}),$$

with strict inequality for some s_{-i}. In this case, we say that strategy s_i' **weakly dominates** strategy s_i. A strategy is a **weakly dominant strategy** for player i in game $\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]$ if it weakly dominates every other strategy in S_i.

Definition 8.B.4: A strategy $\sigma_i \in \Delta(S_i)$ is **strictly dominated** for player i in game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$ if there exists another strategy $\sigma_i' \in \Delta(S_i)$ such that for all $\sigma_{-i} \in \prod_{j \neq i} \Delta(S_j)$,

$$u_i(\sigma_i', \sigma_{-i}) > u_i(\sigma_i, \sigma_{-i}).$$

In this case, we say that strategy σ_i' **strictly dominates** strategy σ_i. A strategy σ_i is a **strictly dominant strategy** for player i in game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$ if it strictly dominates every other strategy in $\Delta(S_i)$.

Proposition 8.B.1: Player i's pure strategy $s_i \in S_i$ is strictly dominated in game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$ if and only if there exists another strategy $\sigma_i' \in \Delta(S_i)$ such that

$$u_i(\sigma_i', s_{-i}) \geq u_i(s_i, s_{-i})$$

for all $s_{-i} \in S_{-i}$.

Definition 8.C.1: In game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$, strategy σ_i is a **best response** for player i to his rivals strategies σ_{-i} if

$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i})$$

for all $\sigma_i' \in \Delta(S_i)$. Strategy σ_i is **never a best response** if there is no σ_{-i} for which σ_i is a best response.

Definition 8.C.2: In game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$, the strategies in $\Delta(S_i)$ that survive the iterated removal of strategies that are never a best response are known as player i's **rationalizable strategies**.

Definition 8.D.1: A strategy profile $s = (s_1, \ldots, s_I)$ constitutes a **Nash equilibrium** of game $\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]$ if for every $i = 1, \ldots, I$,

$$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$$

for all $s_i' \in S_i$.

Definition 8.D.2: A mixed strategy profile $\sigma = (\sigma_1, \ldots, \sigma_I)$ constitutes a **Nash equilibrium** of game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$ if for every $i = 1, \ldots, I$,

$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i})$$

for all $\sigma_i' \in \Delta(S_i)$.

Proposition 8.D.1: Let $S_i^+ \subset S_i$ denote the set of pure strategies that player i plays with positive probability in mixed strategy profile $\sigma = (\sigma_1, \ldots, \sigma_I)$. Strategy profile σ is a Nash equilibrium in game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$ if and only if for all $i = 1, \ldots, I$,
(i) $u_i(s_i, \sigma_{-i}) = u_i(s_i', \sigma_{-i})$ for all $s_i, s_i' \in S_i^+$,
(ii) $u_i(s_i, \sigma_{-i}) \geq u_i(s_i', \sigma_{-i})$ for all $s_i \in S_i^+$ and $s_i' \notin S_i^+$.

Corollary 8.D.1: Pure strategy profile $s = (s_1, \ldots, s_I)$ is a Nash equilibrium of game $\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]$ if and only if it is a (degenerate) mixed strategy Nash equilibrium of game $\Gamma_N' = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$.

Proposition 8.D.2: Every game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$ in which the sets $S_1, \ldots, S_I$ have a finite number of elements has a mixed strategy Nash equilibrium.

Proposition 8.D.3: A Nash equilibrium exists in game $\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]$ if for all $i = 1, \ldots, I$,
(i) S_i is a nonempty, convex, and compact subset of some Euclidean space $\mathbb{R}^M$.
(ii) $u_i(s_1, \ldots, s_I)$ is continuous in $(s_1, \ldots, s_I)$ and quasiconcave in s_i.

Definition 8.E.1: A (pure strategy) **Bayesian Nash equilibrium** for the Bayesian game $[I, \{S_i\}, \{u_i(\cdot)\}, \Theta, F(\cdot)]$ is a profile of decision rules $(s_1(\cdot), \ldots, s_I(\cdot))$ that constitutes a Nash equilibrium of game $\Gamma_N = [I, \{\mathscr{S}_i\}, \{\tilde{u}_i(\cdot)\}]$. That is, for every $i = 1, \ldots, I$,

$$\tilde{u}_i\big(s_i(\cdot), s_{-i}(\cdot)\big) \geq \tilde{u}_i\big(s_i'(\cdot), s_{-i}(\cdot)\big)$$

for all $s_i'(\cdot) \in \mathscr{S}_i$, where

$$\tilde{u}_i\big(s_1(\cdot), \ldots, s_I(\cdot)\big) = \mathbb{E}_\theta\Big[u_i\big(s_1(\theta_1), \ldots, s_I(\theta_I), \theta_i\big)\Big].$$

Proposition 8.E.1: A profile of decision rules $(s_1(\cdot), \ldots, s_I(\cdot))$ is a Bayesian Nash equilibrium in Bayesian game $[I, \{S_i\}, \{u_i(\cdot)\}, \Theta, F(\cdot)]$ if and only if, for all i and all $\bar{\theta}_i \in \Theta_i$ occurring with positive probability

$$\mathbb{E}_{\theta_{-i}}\left[u_i\left(s_i\left(\bar{\theta}_i\right), s_{-i}(\theta_{-i}), \bar{\theta}_i\right) | \bar{\theta}_i\right] \geq \mathbb{E}_{\theta_{-i}}\left[u_i\left(s_i', s_{-i}(\theta_{-i}), \bar{\theta}_i\right) | \bar{\theta}_i\right]$$

for all $s_i' \in S_i$, where the expectation is taken over realizations of the other players' random variables conditional on player i's realization of his sigmal $\bar{\theta}_i$.

Definition: For any normal form game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$, a **perturbed game** is $\Gamma_\epsilon = [I, \{\Delta_\epsilon(S_i)\}, \{u_i(\cdot)\}]$ in which player i's strategy set is

$$\Delta_\epsilon(S_i) = \left\{\sigma_i : \sigma_i(s_i) \geq \epsilon_i(s_i) \text{ for all } s_i \in S_i \text{ and } \sum_{s_i \in S_i} \sigma_i(s_i) = 1\right\},$$

with $\epsilon(s_i) \in (0, 1)$ and $\sum_{s_i \in S_i} \epsilon_i(s_i) < 1$. That is, perturbed game Γ_ϵ is derived from the original game Γ_N by requiring that each player i play every one of his strategies, say s_i, with at least some minimal positive probabilities $\epsilon_i(s_i)$.

Definition 8.F.1: A Nash equilibrium σ of game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$ is **(normal form) trembling-hand perfect** if there is some sequence of perturbed games $\{\Gamma_{\epsilon^k}\}_{k=1}^\infty$ that converges to Γ_N, for which there is some associated sequence of Nash equilibria $\{\sigma^k\}_{k=1}^\infty$ that converges to σ.

Proposition 8.F.1: A Nash equilibrium σ of game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$ is (normal form) trembling hand perfect if and only if there is some sequence of totally mixed strategies $\{\sigma^k\}_{k=1}^\infty$ such that $\lim_{k\to\infty} \sigma^k = \sigma$ and σ_i is a best response to every element of sequence $\{\sigma_{-i}^k\}_{k=1}^\infty$ for all $i = 1, \ldots, I$.

Proposition 8.F.2: If $\sigma = (\sigma_1, \ldots, \sigma_I)$ is a (normal form) trembling-hand perfect Nash equilibrium, then σ_i is not a weakly dominated strategy for any $i = 1, \ldots, I$. Hence, in any (normal form) trembling-hand perfect Nash equilibrium, no weakly dominated pure strategy can be played with positive probability.

Lemma 8.AA.1: If the sets $S_1, \cdots, S_I$ are nonempty, S_i is compact and convex, and $u_i(\cdot)$ is continuous in $(s_1, \cdots, s_I)$ and quasiconcave in s_i, then player i's best-response correspondence $b_i(\cdot)$ is nonempty, convex-valued, and upper hemicontinuous.

Proposition 9.B.1: (**Zermelo's Theorem**) Every finite game of perfect information Γ_E has a pure strategy Nash equilibrium that can be derived through backward induction. Moreover, if no player has the same payoffs at any two terminal nodes, then there is a unique Nash equilibrium that can be derived in this manner.

Definition 9.B.1: A **subgame** of an extensive form game Γ_E is a subset of the game having the following properties:
(i) It begins with an information set containing a single decision node, contains all the decision nodes that are successors (both immediate and later) of this node, and contains *only* those nodes.
(ii) If decision node x is in the subgame, then every $x' \in H(x)$ is also, where $H(x)$ is the information set that contains decision node x. (That is, there are no "broken" information sets.)

Definition 9.B.2: A profile of strategies $\sigma = (\sigma_1, \dots \sigma_I)$ in an I-player extensive form game Γ_E is a **subgame perfect Nash equilibrium (SPNE)** if it induces a Nash equilibrium in every subgame of Γ_E.

Proposition 9.B.2: Every finite game of perfect information Γ_E has a pure strategy subgame perfect Nash equilibrium. Moreover, if no player has the same payoffs at any two terminal nodes, then there is a unique subgame perfect Nash equilibrium.

Proposition 9.B.3: Consider an extensive form game Γ_E and some subgame S of Γ_E. Suppose that strategy profile σ^S is an SPNE in subgame S, and let $\widehat{\Gamma}_E$ be the reduced game formed by replacing subgame S by a terminal node with payoffs equal to those arising from play of σ^S.
(i) In any SPNE σ of Γ_E in which σ^S is the play in subgame S, players' moves at information sets outside subgame S must constitute an SPNE of reduced game $\widehat{\Gamma}_E$.
(ii) If $\hat{\sigma}$ is an SPNE of $\widehat{\Gamma}_E$, then the strategy profile σ that specifies the moves in σ^S at information sets in subgame S and that specifies the moves in $\hat{\sigma}$ at information sets not in S is an SPNE of Γ_E.

Proposition 9.B.4: Consider an I-player extensive form game Γ_E involving successive play of T simultaneous-move games, $\Gamma_N^t = \left[I, \{\Delta(S_i^t)\}, \{u_i^t\}\right]$ for $t = 1, \dots, T$, with the players observing the pure strategies played in each game immediately after its play is concluded. Assume that each player's payoff is equal to the sum of her payoffs in the plays of the T games. If there is a unique Nash equilibrium in each game Γ_N^t, say $\sigma^t = (\sigma_1^t, \dots, \sigma_I^t)$, then there is a unique SPNE of Γ_E and it consists of each player i playing strategy σ_i^t in each game Γ_N^t regardless of what has happened previously.

Definition 9.C.1: A **system of beliefs** μ in extensive form game Γ_E is a specification of a probability $\mu(x) \in [0,1]$ for each decision node x in Γ_E such that $\sum_{x \in H} \mu(x) = 1$ for all information sets H.

Definition 9.C.2: A strategy profile $\sigma = (\sigma_1, \ldots, \sigma_I)$ in extensive form game Γ_E is **sequentially rational at information set H given a system of beliefs μ** if, denoting by $\iota(H)$ the player who moves at information set H, we have

$$\mathbb{E}[u_{\iota(H)}|H, \mu, \sigma_{\iota(H)}, \sigma_{-\iota(H)}] \geq \mathbb{E}[u_{\iota(H)}|H, \mu, \tilde{\sigma}_{\iota(H)}, \sigma_{-\iota(H)}]$$

for all $\tilde{\sigma}_{\iota(H)} \in \Delta(S_{\iota(H)})$. If strategy profile σ satisfies this condition for all information sets H, then we say that σ is **sequentially rational given belief system μ**.

Definition 9.C.3: A profile of strategies and system of beliefs (σ, μ) is a **weak perfect Bayesian equilibrium (WPBE)** in extensive form game Γ_E if it has the following properties:
(i) The strategy profile σ is sequentially rational given belief system μ.
(ii) The system of beliefs μ is derived from strategy profile σ through Bayes' rule whenever possible. That is, for any information set H such that $Pr(H|\sigma) > 0$, we must have

$$\mu(x) = \frac{Pr(x|\sigma)}{Pr(H|\sigma)} \text{ for all } x \in H.$$

Definition 9.C.4: A strategy profile and system of beliefs (σ, μ) is a **sequential equilibrium** of extensive form game Γ_E if it has the following properties:
(i) Strategy profile σ is sequentially rational given belief system μ.
(ii) There exists a sequence of completely mixed strategies σ_k with limit σ such that the sequence of beliefs $\{\mu_k\}$ derived from $\{\sigma_k\}$ approaches μ as $k \longrightarrow \infty$.

Proposition 9.C.2: In every sequential equilibrium (σ, μ) of an extensive form game Γ_E, the equilibrium strategy profile σ constitutes a subgame perfect Nash equilibrium of Γ_E.

Definition 9.BB.1: Strategy profile σ in extensive form game Γ_E is an **extensive form trembling-hand perfect Nash equilibrium** if and only if it is a normal form trembling-hand perfect Nash equilibrium of the agent normal form derived from Γ_E.

Proposition 12.C.1: There is a unique Nash equilibrium (p_1^*, p_2^*) in the Bertrand duopoly model. In this equilibrium, both firms set their prices equal to cost: $p_1^* = p_2^* = c$.

Proposition 12.C.2: In any Nash equilibrium of the Cournot duopoly model with cost $c > 0$ per unit for the two firms and an inverse demand function $p(\cdot)$ satisfying $p'(q) < 0$ for all $q \geq 0$ and $p(0) > c$, the market price is greater than c (the competitive price) and smaller than the monopoly price.

Proposition 12.D.1: Consider the following strategies for firms $j = 1, 2$:

$$p_{jt}(H_{t-1}) = \begin{cases} p^m & \text{if all elements of } H_{t-1} \text{ equal } (p^m, p^m) \text{ or } t = 1 \\ c & \text{otherwise.} \end{cases}$$

This strategy profile constitutes a subgame perfect Nash equilibrium (SPNE) of the infinitely repeated Bertrand duopoly game if and only if $\delta \geq \frac{1}{2}$.

Proposition 12.D.2: In the infinitely repeated Bertrand duopoly game, when $\delta \geq \frac{1}{2}$, repeated choice of any price $p \in [c, p^m]$ can be supported as a subgame perfect Nash equilibrium outcome path using Nash reversion strategies. By contrast, when $\delta \leq \frac{1}{2}$, any subgame perfect Nash equilibrium outcome path must have all sales occurring at a price equal to c in every period.

Proposition 12.E.1: Suppose that conditions

(A1) $Jq_J \geq J'q_{J'}$ whenever $J > J'$,

(A2) $q_J \leq q_{J'}$ whenever $J > J'$,

(A3) $p(Jq_J) - c'(q_J) \geq 0$ for all J

are satisfied by the post-entry oligopoly game, that $p'(\cdot) < 0$, and that $c''(\cdot) \geq 0$. Then the equilibrium number of entrants, J^*, is at least $J^\circ - 1$, where J° is the socially optimal number of entrants.

Proposition 12.F.1: As the market size grows, the price in any subgame perfect Nash equilibrium of the two-stage Cournot entry model converges to the level of minimum average cost (the "competitive" price). Formally,

$$\max_{p_\alpha \in P_\alpha} |p_\alpha - \bar{c}| \longrightarrow 0 \quad \text{as} \quad \alpha \longrightarrow \infty.$$

Definition 12.AA.1: A strategy profile $s = (s_1, s_2)$ in an infinitely repeated game is one of **Nash reversion** if each player's strategy calls for playing some outcome path Q until someone defects and playing the stage game Nash equilibrium $q^* = (q_1^*, q_2^*)$ thereafter.

Lemma 12.AA.1: Nash reversion strategy profile that calls for playing outcome path $Q = \{q_{1t}, q_{2t}\}_{t=1}^{\infty}$ prior to any deviation is an SPNE if and only if

$$\hat{\pi}_i(q_{jt}) + \frac{\delta}{1-\delta}\pi_i(q_1^*, q_2^*) \leq v_i(Q, t)$$

where $j \neq i$, for all t and $i = 1, 2$.

Proposition 12.AA.1: Consider an infinitely repeated game with $\delta > 0$ and $S_i \subset \mathbb{R}$ for $i = 1, 2$. Suppose also that $\pi_i(q)$ is differentiable at $q^* = (q_1^*, q_2^*)$, with $\partial\pi_i(q_i^*, q_j^*)/\partial q_j \neq 0$ for $j \neq i$ and $i = 1, 2$. Then there is some $q' = (q_1', q_2')$, with $[\pi_1(q'), \pi_2(q')] \gg [\pi_1(q^*), \pi_2(q^*)]$ whose infinite repetition is the outcome path of an SPNE that uses Nash reversion.

Proposition 12.AA.2: Suppose that outcome path Q can be sustained as an SPNE outcome path using Nash reversion when the discount rate is δ. Then it can be so sustained for any $\delta' \geq \delta$.

Proposition 12.AA.3: For any pair of actions $q = (q_1, q_2)$ such that $\pi_i(q_1, q_2) > \pi_i(q_1^*, q_2^*)$ for $i = 1, 2$ there exists a $\underline{\delta} < 1$ such that, for all $\delta > \underline{\delta}$, infinite repetition of $q = (q_1, q_2)$ is the outcome path of an SPNE using Nash reversion strategies.

Definition: Player i's **minimax payoff** $\underline{\pi}_i$ is the lowest payoff that player i's rival can hold him to in the stage game if player i anticipates the action that his rival will play. That is,

$$\underline{\pi}_i = \min_{q_{-i}} \left[\max_{q_i} \pi_i(q_i, q_{-i})) \right].$$

Payoffs that strictly exceed $\underline{\pi}_i$ for each player i are known as **individually rational payoffs.**

Proposition 12.AA.4: Consider infinitely repeated game with $\delta > 0$ and $S_i \subset \mathbb{R}$ for $i = 1, 2$. Suppose also that $\pi_i(q)$ is differentiable at $q^* = (q_1^*, q_2^*)$, with $\partial\pi_i(q_i^*, q_j^*)/\partial q_j \neq 0$ for $j \neq i$ and $i = 1, 2$, and that $\pi_i(q_1^*, q_2^*) > \underline{\pi}_i$ for $i = 1, 2$. Then there is some SPNE with discounted payoffs to the two players of (v_1', v_2') such that $(1-\delta)v_i' < \pi_i(q_1^*, q_2^*)$ for $i = 1, 2$.

Proposition 12.AA.5: (The Folk Theorem) For any feasible pair of individually rational payoffs $(\pi_1, \pi_2) \gg (\underline{\pi}_1, \underline{\pi}_2)$, there exists a $\underline{\delta} < 1$ such that, for all $\delta > \underline{\delta}$, (π_1, π_2) are the average payoffs arising in an SPNE.

Definition 13.B.1: In the competitive labor market model with unobservable worker productivity levels, a **competitive equilibrium** is a wage rate w^* and a set Θ^* of worker types who accept employment such that

$$\Theta^* = \{\theta : r(\theta) \leq w^*\}$$

and

$$w^* = \mathbb{E}[\theta|\theta \in \Theta^*].$$

Proposition 13.B.1: Let W^* denote the set of competitive equilibrium wages for the adverse selection labor market model, and let $w^* = \max\{w : w \in W^*\}$.
(i) If $w^* > r(\underline{\theta})$ and there is an $\varepsilon > 0$ such that $\mathbb{E}[\theta|r(\theta) \leq w'] > w'$ for all $w' \in (w^* - \varepsilon, w^*)$, then there is a unique pure strategy SPNE of the two-stage game-theoretic model. In this SPNE, employed workers receive a wage of w^*, and workers with types in the set $\Theta(w^*) = \{\theta : r(\theta) \leq w^*\}$ accept employment in firms.
(ii) If $w^* = r(\underline{\theta})$, then there are multiple pure strategy SPNEs. However, in every pure strategy SPNE each agent's payoff exactly equals her payoff in the highest-wage competitive equilibrium.

Definition: An allocation that cannot be Pareto improved by an authority who is unable to observe agents' private information is known as a **constrained (or second-best) Pareto optimum**.

Proposition 13.B.2: In the adverse selection labor market model (where $r(\cdot)$ is strictly increasing, with $r(\theta) \leq \theta$ for all $\theta \in [\underline{\theta}, \bar{\theta}]$, and $F(\cdot)$ has an associated density $f(\cdot)$, with $f(\theta) > 0$ for all $\theta \in [\underline{\theta}, \bar{\theta}]$), the highest-wage competitive equilibrium is a constrained Pareto optimum.

Definition: (Signalling game) Let $u(w, e|\theta)$ denote the utility of a type θ worker who chooses educational level e and receives wage w. Let

$$u(w, e|\theta) = w - c(e, \theta).$$

The **signalling game** proceeds as follows: initially, a random move of nature determines whether worker is of high or low ability. Then, conditional on her type the worker chooses how much education to obtain. After obtaining her chosen education level, the worker enters the job market. Conditional on the observed education level of the worker, two firms simultaneously make wage offers to her. Finally, the worker decides whether to work for a firm and, if so, which one.

Definition: The **single-crossing property** is the property that the slope of the indifference curve of the higher type is smaller. That is, $c_{e\theta}(e, \theta) < 0$.

Lemma 13.C.1: In any separating perfect Bayesian equilibrium, $w^*(e^*(\theta_H)) = \theta_H$ and $w^*(e^*(\theta_L)) = \theta_L$; that is, each worker type receives a wage equal to her productivity level.

Lemma 13.C.2: In any separating perfect Bayesian equilibrium, $e^*(\theta_L) = 0$; that is, a low-ability worker chooses to get no education.

Definition: (Screening game) Assume that the utility of a type θ worker who receives wage w and faces task level $t \geq 0$ is

$$u(w, t|\theta) = w - c(t, \theta),$$

where $c(0, \theta) = 0$, $c_t(t, \theta) > 0$, $c_t t(t, \theta) > 0$, $c_\theta(t, \theta) < 0$ for all $t > 0$, and $c_{t\theta}(t, \theta) < 0$. A **screening game** is the following two-stage game.

- Stage 1: Two firms simultaneously announce sets of offered contracts. A **contract** is a pair (w, t). Each firm may announce any finite number of contracts.
- Stage 2: Given the offers made by the firms, workers of each type choose whether to accept a contract and, if so, which one. If a worker's most preferred contract is offered by both firms, she accepts each firm's offer with probability $\frac{1}{2}$.

Proposition 13.D.1: In any SPNE of the screening game with observable worker types, a type θ_i worker accepts contract $(w_i^*, t_i^*) = (\theta_i, 0)$, and firms earn zero profits.

Lemma 13.D.1: In any equilibrium, whether pooling or separating, both firms must earn zero profits.

Lemma 13.D.2: No pooling equilibria exist.

Lemma 13.D.3: If (w_L, t_L) and (w_H, t_H) are the contracts signed by the low- and high-ability workers in a separating equilibrium, then both contracts yield zero profits; that is, $w_L = \theta_L$ and $w_H = \theta_H$.

Lemma 13.D.4: In any separating equilibrium, the low-ability workers accept contract $(\theta_L, 0)$; that is, they receive the same contract as when no informational imperfections are present in the market.

Lemma 13.D.5: In any separating equilibrium, the high-ability workers accept contract $(\theta_H, \hat{t}_H)$, where $\hat{t}_H$ satisfies $\theta_H - c(\hat{t}_H, \theta_L) = \theta_L - c(0, \theta_L)$.

Proposition 13.D.2: In any subgame perfect Nash equilibrium of the screening game, low-ability workers accept contract $(\theta_L, 0)$, and high-ability workers accept contract $(\theta_H, \hat{t}_H)$, where $\hat{t}_H$ satisfies $\theta_H - c(\hat{t}_H, \theta_L) = \theta_L - c(0, \theta_L)$.

Definition: Denote the equilibrium payoff to type θ in PBE $(a^*(\theta), s^*(a), \mu)$ by

$$u_1^*(\theta) = u_1(a^*(\theta), s^*(a^*(\theta)), \theta).$$

We then say that action a is **equilibrium dominated** for type θ in PBE $(a^*(\theta), s^*(a), \mu)$ if

$$u_1^*(\theta) > \max_{s \in S^*(\Theta, a)} u_1(a, s, \theta).$$

Using this notion of dominance, define for each $a \in A$ the set $\Theta^{**}(a)$ of θ such that the above condition does not hold. We can now say that a PBE has reasonable beliefs if for all actions a with $\Theta^{**}(a) \neq \varnothing$, $\mu(\theta|a) > 0$ only if $\theta \in \Theta^{**}(a)$.

Definition: In signaling games with two types, the equilibrium dominance-based refinement is equivalent to the **intuitive criterion**. A PBE is said to violate the intuitive criterion if there exists a type θ and an action $a \in A$ such that

$$\min_{s \in S^*(\Theta^{**}(a), a)} u_1(a, s, \theta) > u_1^*(\theta).$$

CHAPTER 14. THE PRINCIPAL-AGENT PROBLEM

Proposition 14.B.1: In the principal-agent model with observable managerial effort, an optimal contract specifies that the manager choose the effort e^* that maximizes

$$\int \pi f(\pi|e) d\pi - v^{-1}(\bar{u} + g(e)),$$

and pays the manager a fixed wage

$$w^* = v^{-1}(\bar{u} + g(e^*)).$$

This is the uniquely optimal contract if $v''(w) < 0$ at all w.

Proposition 14.B.2: In the principal-agent model with unobservable managerial effort and a risk-neutral manager, an optimal contract generates the same effort choice and expected utilities for the manager and the owner as when effort is observable.

Definition: In the principal-agent model with unobservable managerial effort e, an **optimal contract** for implementing a specific effort level e solves

$$\min_{w(\pi)} \int w(\pi) f(\pi|e) d\pi$$

subject to the **participation (individual rationality) constraint**

$$\int v\left(w(\pi)\right) f(\pi|e) d\pi - g(e) \geq \bar{u},$$

and the **incentive compatibility constraint**

$$e \quad \text{solves} \quad \max_{\tilde{e}} \int v\left(w(\pi)\right) f(\pi|\tilde{e}) d\pi - g(\tilde{e}).$$

Lemma 14.B.1: In any solution to the optimal contract problem with $e = e_H$, both participation and incentive compatibility constraints bind: $\gamma > 0$ and $\mu > 0$.

Proposition 14.B.3: In the principal-agent model with unobservable manager effort, a risk-averse manager, and two possible effort choices, the optimal compensation scheme for implementing e_H satisfies condition

$$\frac{1}{v'(w(\pi))} = \gamma + \mu \left[1 - \frac{f(\pi, y|e_L)}{f(\pi, y|e_H)}\right],$$

gives the manager expected utility $\bar{u}$, and involves a larger expected wage payment than is required when effort is observable. The optimal compensation scheme for implementing e_L involves the same fixed wage payment as if effort were observable. Whenever the optimal effort level with observable effort would be e_H, nonobservability causes a welfare loss.

Proposition 14.C.1: In the principal-agent model with an observable state variable θ, the optimal contract involves an effort level e_i^* in state θ_i such that $\pi'(e_i^*) = g_e(e_i^*, \theta_i)$ and fully insures the manager, setting his wage in each state θ_i at the level w_i^* such that $v(w_i^* - g(e_i^*, \theta_i)) = \bar{u}$.

Proposition 14.C.2: (The Revelation Principle) Denote the set of possible states by Θ. In searching for an optimal contract, the owner can without loss restrict himself to contracts of the following form:

(i) After the state θ is realized, the manager is required to announce which state has occurred.

(ii) The contract specifies an outcome $\left(w(\hat{\theta}), e(\hat{\theta})\right)$ for each possible announcement $\hat{\theta} \in \Theta$.

(iii) In every state $\theta \in \Theta$, the manager finds it optimal to report the state truthfully.

Definition: In the principal-agent model with unobservable state variable θ and an infinitely risk-averse manager, an **optimal contract** solves

$$\max_{w_H, e_H \geq 0, w_L, e_L \geq 0} \lambda \left(\pi(e_H) - w_H\right) + (1 - \lambda) \left(\pi(e_L) - w_)\right),$$

subject to the **reservation utility** (or **individual rationality**) **constraints**

$$w_L - g(e_L, \theta_L) \geq v^{-1}(\bar{u}),$$
$$w_H - g(e_H, \theta_H) \geq v^{-1}(\bar{u}),$$

and the **incentive compatibility** (or **truth-telling** or **self-selection**) **constraints**

$$w_H - g(e_H, \theta_H) \geq w_L - g(e_L, \theta_H),$$
$$w_L - g(e_L, \theta_L) \geq w_H - g(e_H, \theta_L).$$

Lemma 14.C.1: We can ignore the individual rationality constraint for the high type. That is, a contract is a solution to the optimal contract if and only if it is the solution to the problem derived from the optimal contract problem by dropping this constraint.

Lemma 14.C.2: An optimal contract must have $w_L - g(e_L,) = v^{-1}(\bar{u})$.

Lemma 14.C.3: In any optimal contract:
(i) $e_L \leq e_L^*$; that is, the manager's effort level in state θ_L is no more than the level that would arise if θ were observable.
(ii) $e_H = e_H^*$; that is, the manager's effort level in state θ_H is exactly equal to the level that would arise if θ were observable.

Lemma 14.C.4: In any optimal contract, $e_L < e_L^*$; that is, the effort level in state θ_L is necessarily strictly below the level that would arise in state θ_L if θ were observable.

Proposition 14.C.3: In the hidden information principal-agent model with an infinitely risk-averse manager, the optimal contract sets the level of effort in state θ_H at its first-best (full observability) level e_H^*. The effort level in state θ_L is distorted downward from its first-best level e_L^*. In addition, the manager is inefficiently insured, receiving a utility greater than $\bar{u}$ in state θ_H and a utility equal to $\bar{u}$ in state θ_L. The owner's expected payoff is strictly lower than the expected payoff he receives when θ is observable, while the infinitely risk-averse manager's expected utility is the same as when θ is observable (it equals $\bar{u}$).

CHAPTER 16. EQUILIBRIUM AND ITS BASIC WELFARE PROPERTIES

Definition 16.B.1: An **allocation** $(x, y) = (x_1, \ldots, x_I, y_1, \ldots, y_J)$ is a specification of a consumption vector $x_i \in X$ for each consumer $i = 1, \ldots, I$ and a production vector $y_i \in Y_i$ for each firm $j = 1, \ldots, J$. An allocation (x, y) is (x, y) **feasible** if $\sum_i x_{li} = \bar{\omega}_l + \sum_j y_{lj}$ for every commodity l. That is, if

$$\sum_i x_i = \bar{\omega} + \sum_j y_j.$$

Definition 16.B.2: A feasible allocation (x, y) is **Pareto optimal** (or **Pareto efficient**) if there is no other allocation $(x', y') \in A$ that **Pareto dominates** it, that is, if there is no feasible allocation (x', y') such that $x_i' \succsim_i x_i$ for all i and $x_i' \succ_i x_i$ for some i.

Definition 16.B.3: Given a private ownership economy specified by

$$\left(\{(X_i, \succsim_i)\}_{i=1}^{I}, \{Y_j\}_{j=1}^{J}, \{(\omega_i, \theta_{i1}, \ldots, \theta_{iJ})\}_{i=1}^{I} \right),$$

an allocation (x^*, y^*) and a price vector $p = (p_1, \ldots, p_L)$ constitute a **Walrasian** (or **competitive**) **equilibrium** if:

(i) For every j, y_j^* maximizes profits in Y_j; that is,

$$p \cdot y_j \leq p \cdot y_j^* \quad \text{for all } y_j \in Y_j.$$

(ii) For every i, x_i^* is maximal for $\succsim_i$ in the budget set

$$\left\{x_i \in X_i : p \cdot x_i \leq p \cdot \omega_i + \sum_j \theta_{ij} p \cdot y_j^*\right\}.$$

(iii) $\sum_i x_i^* = \bar{\omega} + \sum_j y_j^*$.

Definition 16.B.4: Given an economy specified by $\left(\{(X_i, \succsim_i)\}_{i=1}^I, \{Y_j\}_{j=1}^J, \bar{\omega}\right)$ an allocation (x^*, y^*) and a price vector $p = (p_1, \ldots, p_L)$ constitute a **price equilibrium with transfers** if there is an assignment of wealth levels $(w_1, \ldots, w_I)$ with $\sum_i w_i = p \cdot \bar{\omega} + \sum_j p \cdot y_j^*$ such that

(i) For every j, y_j^* maximizes profits in Y_j; that is,

$$p \cdot y_j \leq p \cdot y_j^* \quad \text{for all } y_j \in Y_j.$$

(ii) For every i, x_i^* is maximal for $\succsim_i$ in the budget set

$$\left\{x_i \in X_i : p \cdot x_i \leq w_i\right\}.$$

(iii) $\sum_i x_i^* = \bar{\omega} + \sum_j y_j^*$.

Definition 16.C.1: The preference relation $\succsim_i$ on the consumption set X_i is **locally nonsatiated** if for every $x_i \in X_i$ and every $\varepsilon > 0$, there is an $x_i' \in X_i$ such that $\| x_i' - x_i \| \leq \varepsilon$ and $x_i' \succ_i x_i$.

Proposition 16.C.1: (**First Fundamental Theorem of Welfare Economics**) If preferences are locally nonsatiated, and if (x^*, y^*, p) is a price equilibrium with transfers, then the allocation (x^*, y^*) is Pareto optimal. In particular, any Walrasian equilibrium allocation is Pareto optimal.

Definition 16.D.1: Given an economy specified by $\left(\{(X_i, \succsim_i)\}_{i=1}^I, \{Y_j\}_{j=1}^J, \bar{\omega}\right)$ an allocation (x^*, y^*) and a price vector $p = (p_1, \ldots, p_I) \neq 0$ constitute a **price quasiequilibrium with transfers** if there is an assignment of wealth levels $(w_1, \ldots, w_I)$ with $\sum_i w_i = p \cdot \bar{\omega} + \sum_j p \cdot y_j^*$ such that

(i) For every j, y_j^* maximizes profits in Y_j; that is,

$$p \cdot y_j \leq p \cdot y_j^* \quad \text{for all } y_j \in Y_j.$$

(ii) For every i, if $x_i \succ_i x_i^*$ then $p \cdot x_i \geq w_i$.

(iii) $\sum_i x_i^* = \bar{\omega} + \sum_j y_j^*$.

Proposition 16.D.1: (**Second Fundamental Theorem of Welfare Economics**) Consider an economy specified by $\left(\{(X_i, \succsim_i)\}_{i=1}^I, \{Y_j\}_{j=1}^J, \bar{\omega}\right)$, and suppose that every Y_i is convex and every preference relation $\succsim_i$ is convex [i.e., the set $\{x'_i \in X_i : x'_i \succsim_i x_i\}$ is convex for every $x_i \in X_i$] and locally nonsatiated. Then, for every Pareto optimal allocation (x^*, y^*), there is a price vector $p = (p_1, \ldots, p_L) \neq 0$ such that (x^*, y^*, p) is a price quasiequilibrium with transfers.

Proposition 16.D.2: Assume that X_i is convex and $\succsim_i$ is continuous. Suppose also that the consumption vector $x_i^* \in X_i$, the price vector p, and the wealth level w_i are such that $x_i \succ_i x_i^*$ implies $p \cdot x_i \geq w_i$. Then, if there is a consumption vector $x'_i \in X_i$ such that $p \cdot x'_i < w_i$ [a cheaper consumption for (p, w_i)], it follows that $x_i \succ_i x_i^*$ implies $p \cdot x_i > w_i$.

Proposition 16.D.3: Suppose that for every i, X_i is convex, $0 \in X_i$, and $\succsim_i$ is continuous. Then any price quasiequilibrium with transfer that has $(w_1, \ldots, w_I) \gg 0$ is a price equilibrium with transfers.

Proposition 16.E.1: A feasible allocation $(x, y) = (x_1, \ldots, x_I, y_1, \ldots, y_J)$ is a Pareto optimum if and only if $\big(u_1(x_1), \ldots, u_I(x_I)\big) \in UP$, where UP is defined as the **Pareto frontier** of the **utility possibility set** U.

Proposition 16.E.2: if $u^* = (u_1^*, \ldots, u_I^*)$ is a solution to the social welfare maximization problem

$$\max_{u \in U} \lambda \cdot u$$

with $\lambda \gg 0$, then $u^* \in UP$; that is, u^* is the utility vector of a Pareto optimal allocation. Moreover, if the **utility possibility set** U is convex, then for any $\tilde{u} = (\tilde{u_1}, \ldots, \tilde{u_I}) \in UP$, there is a vector of welfare weights $\lambda = (\lambda_i, \ldots, \lambda_I) \geq 0$, $\lambda \neq 0$ such that $\lambda \cdot \tilde{u} \geq \lambda \cdot u$ for all $u \in U$, that is, such that $\tilde{u}$ is a solution to the social welfare maximization problem

$$\max_{u \in U} \lambda \cdot u.$$

Proposition 16.F.1: Under the assumptions made about the economy (in particular, the concavity of every $u_i(\cdot)$ and the convexity of every $F_j(\cdot)$), every Pareto optimal allocation (and, hence, every price equilibrium with transfers) maximizes a weighted sum of utilities subject to the resource and technological constraints. Moreover, the weight λ_i fo the utility of the ith consumer equals the reciprocal of consumer i's marginal utility of wealth evaluated at the supporting prices and imputed wealth.

Definition 16.G.1: A **Lindahl equilibrium** for the public goods economy is a price equilibrium with transfers for the artificial economy with personalized commodities. That is, an allocation $(x_1^*, \ldots, x_I^*, q^*, z^*) \in \mathbb{R}^{2I} \times \mathbb{R} \times \mathbb{R}$ and a price system $(p_1, p_{21}, \ldots, p_{2I}) \in \mathbb{R}^{I+1}$ constitute a

Lindahl equilibrium if there is a set of wealth levels $(w_1, \ldots, w_I)$ satisfying $\sum_i w_i = \sum_i p_1 x_{1i}^* + (\sum_i p_{2i})\, q^* - p_1 z^*$ and such that:
(i) $q^* \leq f(z^*)$ and $(\sum_i p_{2i})q^* - p_1 z^* \geq (\sum_i p_{2i})q - p_1 z$ for all (q, z) with $z \geq 0$ and $q \leq f(z)$.
(ii) For every i, $x_i^* = (x_{1i}^*, x_{2i}^*)$ is maximal for $\succsim_i$ in the set

$$\{(x_{1i}, x_{2i}) \in X_i : p_1 x_{1i} + p_{2i} x_{2i} \leq w_i\}.$$

(iii) $\sum_i x_{1i}^* + z^* = \bar{\omega}_1$ and $x_{2i}^* = q^*$ for every i.

Proposition 16.G.1: Suppose that the basic assumptions of Section 16.F hold and that, in addition, all consumers have convex preferences (so utility functions are quasiconcave). If (x^*, y^*) is Pareto optimal, then there exists a price vector $p = (p_1, \ldots, p_L)$ and wealth levels $(w_1, \ldots, w_I)$ with $\sum_i w_i = p \cdot \bar{\omega} + \sum_j p \cdot y_j^*$ such that:
(i) For any firm j, we have

$$p = \gamma_j \nabla F_j(y_j^*) \quad \text{for some } \gamma_j > 0.$$

(ii) For any i, x_i^* is maximal for $\succsim_i$ in the budget set

$$\{x_i \in X_i : p \cdot x_i \leq w_i\}.$$

(iii) $\sum_i x_i^* = \bar{\omega} + \sum_j y_j^*$.

Proposition 16.AA.1: Suppose that
(i) Every X_i **(a)** is closed; **(b)** is bounded below (i.e., there is $r > 0$ such that $x_{\ell i} > -r$ for every ℓ and i; in words, no consumer can supply to the market an arbitrarily large amount of any good).
(ii) Every Y_j is closed. Moreover, the aggregate production set $Y = \sum_j Y_j$ **(a)** is convex; **(b)** admits the possibility of inaction (i.e., $0 \in Y$); **(c)** satisfies the no-free-lunch property (i.e., $y \geq 0$ and $y \in Y$ implies $y = 0$); **(d)** is irreversible ($y \in Y$ and $-y \in Y$ implies $y = 0$).
Then the set of feasible allocations A is closed and bounded (i.e., there is $r > 0$ such that $|x_{\ell i}| < r$ and $|y_{\ell i}| < r$ for all i, j, ℓ and any $(x, y) \in A$). If, moreover, $-\mathbb{R}_+^L \subset Y$ and we can choose $\hat{x}_i \in X_i$ for every i in such a manner that $\sum_i \hat{x}_i \leq \bar{\omega}$, then A is nonempty.

Proposition 16.AA.2: Suppose that the set of feasible allocations A is nonempty, closed, and bounded and that utility functions $u_i(\cdot)$ are continuous. Then the utility possibility set U is closed and bounded above.

CHAPTER 19. GENERAL EQUILIBRIUM UNDER UNCERTAINTY

Definition 19.B.1: For every physical commodity $\ell = 1, \ldots, L$ and state $s = 1, \ldots, S$, a unit of **(state-) contingent commodity** ℓs is a title to receive a unit of the physical good ℓ if and only if s occurs. Accordingly, a **(state-) contingent commodity vector** is specified by

$$x = (x_{11}, \ldots, x_{L1}, \ldots, x_{1S}, \ldots, x_{LS}) \in \mathbb{R}^{LS},$$

and is understood as an entitlement to receive the commodity vector $(x_{1s}, \ldots, x_{Ls})$ if state s occurs.

Definition 19.C.1: An allocation

$$(x_1^*, \ldots, x_I^*, y_1^*, \ldots y_J^*) \in X_1 \times \cdots \times X_I \times Y_1 \times \cdots \times Y_J \subset \mathbb{R}^{LS(I+J)}$$

and a system of prices for the contingent commodities $p = (p_{11}, \ldots, p_{LS}) \in \mathbb{R}^{LS}$ constitute an **Arrow-Debreu equilibrium** if:

(i) For every j, y_j^* satisfies $p \cdot y_j^* \geq p \cdot y_j$ for all $y_j \in Y_j$.

(ii) For every i, x_i^* is maximal for $\succsim_i$ in the budget set.

$$\left\{x_i \in X_i \;\;:\;\; p \cdot x_i \leq p \cdot \omega_i + \textstyle\sum_j \theta_{ij} p \cdot y_j^*\right\}.$$

(iii) $\sum_i x_i^* = \sum_j y_j^* + \sum_i \omega_i$

Definition 19.D.1: A collection formed by a price vector $q = (q_1, \ldots, q_S) \in \mathbb{R}^S$ for contingent first good commodities at $t = 0$, a spot price vector $p_s = (p_{1s}, \ldots, p_{Ls}) \in \mathbb{R}^L$ for every s, consumption plans $z_i^* = (z_{1i}^*, \ldots, z_{Si}^*) \in \mathbb{R}^S$ at $t = 0$, and $x_i^* = (x_{1i}^*, \ldots, x_{Si}^*) \in \mathbb{R}^{LS}$ at $t = 1$, for every consumer i constitutes a **Radner equilibrium** if

(i) For every i, the consumption plans z_i^*, x_i^* solve the problem

$$\max_{\substack{(x_{1i}, \ldots, x_{Si}) \in \mathbb{R}_+^{LS} \\ (z_{1i}, \ldots, z_{Si}) \in \mathbb{R}_+^S}} U_i(x_{1i}, \ldots, x_{Si})$$

such that (a) $\sum_s q_s z_{si} \leq 0$, and (b) $p_s \cdot x_{si} \leq p_s \cdot \omega_{si} + p_{1s} z_{si}$ for every s.

(ii) $\sum_i z_{si}^* \leq 0$ and $\sum_i x_{si}^* \leq \sum_i \omega_{si}$ for every s.

Proposition 19.D.1: We have:

(i) Suppose the allocation $x^* \in \mathbb{R}^{LSI}$ and the contingent commodities price vector $(p_1, \ldots, p_S) \in \mathbb{R}_{++}^{LS}$ constitute an Arrow-Debreu equilibrium. Then there are prices $q \in \mathbb{R}_{++}^S$ for contingent first good commodities and consumption plans for these commodities $z^* = (z_1^*, \ldots, z_I^*) \in \mathbb{R}^{SI}$ such that the consumptions plans x^*, z^*, the prices q, and the spot prices $(p_1, \ldots, p_S)$ constitute a Radner equilibrium.

(ii) Conversely, suppose the consumption plans $x^* \in \mathbb{R}^{LSI}$, $z^* \in \mathbb{R}^{SI}$ and prices $q \in \mathbb{R}_{++}^S$, $(p_1, \ldots, p_S) \in \mathbb{R}_{++}^{LS}$ constitute a Radner equilibrium. Then there are multipliers $(\mu_1, \ldots, \mu_S) \in \mathbb{R}_{++}^S$ such that the allocation x^* and the contingent commodities price vector $(\mu_1 p_1, \ldots, \mu_S p_S) \in \mathbb{R}_{++}^{LS}$ constitute an Arrow-Debreu equilibrium. (The multiplier μ_s is interpreted as the value, at $t = 0$, of a dollar at $t = 1$ and state s.)

Definition 19.E.1: A unit of an **asset**, or **security**, is a title to receive an amount r_s of good 1 at date $t = 1$ if state s occurs. An asset is therefore characterized by its **return vector** $r = (r_1, \ldots, r_S) \in \mathbb{R}^S$.

Definition 19.E.2: A collection formed by a price vector $q = (q_1, \ldots, q_K) \in \mathbb{R}^K$ for assets traded at $t = 0$, a spot price vector $p_s = (p_{1s}, \ldots, p_{Ls}) \in \mathbb{R}^L$ for every s, portfolio plans $z_i^* = (z_{1i}^*, \ldots, z_{Ki}^*) \in \mathbb{R}^K$ at $t = 0$ and consumption plans $x_i^* = (x_{1i}^*, \ldots, x_{Si}^*) \in \mathbb{R}^{LS}$ at $t = 1$ for every consumer i constitutes a **Radner equilibrium** if:

(i) For every i, the consumption plans z_i^*, x_i^* solve the problem

$$\max_{\substack{(x_{1i},\ldots,x_{Si}) \in \mathbb{R}_+^{LS} \\ (z_{1i},\ldots,z_{Ki}) \in \mathbb{R}_+^{K}}} U_i(x_{1i}, \ldots, x_{Si})$$

such that (a) $\sum_k q_k z_{ki} \leq 0$, and (b) $p_s \cdot x_{si} \leq p_s \cdot \omega_{si} + \sum_k p_{1s} z_{ki} r_{sk}$ for every s.

(ii) $\sum_i z_{ki}^* \leq 0$ and $\sum_i x_{si}^* \leq \sum_i \omega_{si}$ for every k and s.

Proposition 19.E.1: Assume that every return vector is nonnegative and nonzero; that is, $r_k \geq 0$ and $r_k \neq 0$ for all k. Then, for every (column) vector $q \in \mathbb{R}^k$ of asset prices arising in a Radner equilibrium, we can find multipliers $\mu = (\mu_1, \ldots, \mu_s) \geq 0$, such that $q_k = \sum_s \mu_s r_{sk}$ for all k (in matrix notation, $q^{\mathrm{T}} = \mu \cdot R$).

Definition 19.E.3: An asset structure with an $S \times K$ return matrix R is **complete** if rank $R = S$, that is, if there is some subset of S assets with linearly independent returns.

Proposition 19.E.2: Suppose that the asset structure is complete. Then:

(i) If the consumption plan $x^* = (x_1^*, \ldots, x_I^*) \in \mathbb{R}^{LSI}$ and the price vector

$$(p_1, \ldots, p_S) \in \mathbb{R}_{++}^{LS}$$

constitute an Arrow-Debreu equilibrium, then there are asset prices $q \in \mathbb{R}_{++}^k$ and portfolio plans $z^* = (z_1^*, \ldots, z_I^*) \in \mathbb{R}^{KI}$ such that the consumption plans x^*, portfolio plans z^*, asset prices q, and spot prices $(p_1, \ldots, p_S)$ constitute a Radner equilibrium.

(ii) Conversely, if the consumption plans $x^* \in \mathbb{R}^{LSI}$, portfolio plans $z^* \in \mathbb{R}^{KI}$, and prices $p \in \mathbb{R}_{++}^K$, $(p_1, \ldots, p_S) \in \mathbb{R}_{++}^{LS}$ constitute a Radner equilibrium, then there are multipliers $(\mu_1, \ldots, \mu_S) \in \mathbb{R}_{++}^S$ such that the consumption plans x^* and the contingent commodities price vector $(\mu_1 p_1, \ldots, \mu_S p_S) \in \mathbb{R}^{LS}$ constitute an Arrow-Debreu equilibrium. (The multiplier μ_s is interpreted as the value, at $t = 0$, of a dollar at $t = 1$ and state s; recall that $p_{1s} = 1$.)

Proposition 19.E.3: Suppose that the asset price vector $q \in \mathbb{R}^K$, the spot prices $p = (p_1, \ldots, p_S) \in \mathbb{R}^{LS}$, the consumption plans $x^* = (x_1^*, \ldots, x_I^*) \in \mathbb{R}_+^{LSI}$, and the portfolio plans $(z_1^*, \ldots, z_I^*) \in \mathbb{R}^{KI}$ constitute a Radner equilibrium for an asset structure with $S \times K$ return matrix R. Let R' be the $S \times K'$ return matrix of a second asset strcture. If Range $R' =$ Range R then x^* is still the consumption allocation of a Radner equilibrium in the economy with the second asset structure.

Definition 19.F.1: The asset allocation $(z_1, \ldots, z_I) \in \mathbb{R}^{KI}$ is **constrained Pareto optimal** if it is feasible (i.e., $\sum_i z_i \le 0$) and if there is no other feasible asset allocation $(z'_1, \ldots, z'_I) \in \mathbb{R}^{KI}$ such that

$$U_i^*(z'_1, \ldots, z'_I) \ge U_i^*(z_1, \ldots, z_I) \quad \text{for every } i,$$

with at least one inequality strict.

Proposition 19.F.1: Suppose that there are two periods and only one consumption good in the second period. Then any Radner equilibrium is constrained Pareto optimal in the sense that there is no possible redistribution of assets in the first period that leaves every consumer as well off and at least one consumer strictly better off.

Definition 19.G.1: A set $A \subset \mathbb{R}^S$ of random variables is **spanned** by a given asset structure if every $a \in A$ is in the range of the return matrix R of the asset structure, that is, if every $a \in A$ can be expressed as a linear combination of the available asset returns.

Definition 19.H.1: The signal function $\sigma' : S \longrightarrow \mathbb{R}$ is **at least as informative** as $\sigma : S \longrightarrow \mathbb{R}$ if $\sigma(s) \neq \sigma(s')$ implies $\sigma'(s) \neq \sigma'(s')$ for any pair s, s'. It is **more informative** if, in addition, $\sigma'(s) \neq \sigma'(s')$ for some pair s, s' with $\sigma(s) \neq \sigma(s')$.

Proposition 19.H.1: In the single-consumer problem, if the signal function $\sigma'(\cdot)$ is at least as informative as the signal function $\sigma(\cdot)$, then the ex ante utility derived from $\sigma'(\cdot)$, $\sum_s \pi_{si} u_{si}\left(x_{si}^{\sigma'(\cdot)}\right)$, is at least as large as the ex ante utility derived from $\sigma(\cdot)$, $\sum_s \pi_{si} u_{si}\left(x_{si}^{\sigma(\cdot)}\right)$.

Definition 19.H.2: The price function $p(\cdot)$ is a **rational expectations equilibrium price function** if, for every s, $p(s)$ clears the spot market when every consumer i knows that $s \in E_{p(s),\sigma_i(s)}$ and, therefore, evaluates commodity bundles $x_i \in \mathbb{R}^2$ according to the updated utility function

$$\sum_{s'} \Big(\pi_{s'i} \mid p(s), \sigma_i(s)\Big) u_{s'i}(x_i).$$

CHAPTER 21. SOCIAL CHOICE THEORY

Definition 21.B.1: A **social welfare functional** (or **social welfare aggregator**) is a rule $F(\alpha_1, \ldots, \alpha_I)$ that assigns a social preference,

Definition 21.B.2: The social welfare functional $F(\alpha_1, \ldots, \alpha_I)$ is **Paretian**, or has the **Pareto property**, if it respects unanimity of strict preference on the part of the agents, that is, if $F(1, \ldots, 1) = 1$, and $F(-1, \ldots, -1) = -1$.

Definition 21.B.3: The social welfare functional $F(\alpha_1, \ldots, \alpha_I)$ is **symmetric among agents** (or **anonymous**) if the names of the agents do not matter, that is, if a permutation of preferences across agents does not alter the social preference. Precisely, let $\pi : \{1, \ldots, I\} \longrightarrow \{1, \ldots, I\}$ be an onto function (i.e., a function with the property that for any i there is h such that $\pi(h) = i$). Then for any profile $(\alpha_1, \ldots, \alpha_I)$ we have $F(\alpha_1, \ldots, \alpha_I) = F(\alpha_{\pi(1)}, \ldots, \alpha_{\pi(I)})$.

Definition 21.B.4: The social welfare functional $F(\alpha_1, \ldots, \alpha_I)$ is **neutral between alternatives** if $F(\alpha_1, \ldots, \alpha_I) = -F(-\alpha_1, \ldots, -\alpha_I)$ for every profile $(\alpha_1, \ldots, \alpha_I)$, that is, if the social preference is reversed when we reverse the preferences of all agents.

Definition 21.B.5: The social welfare functional $F(\alpha_1, \ldots, \alpha_I)$ is **positively responsive** if, whenever $(\alpha_1, \ldots, \alpha_I) \geq (\alpha'_1, \ldots, \alpha'_I)$, $(\alpha_1, \ldots, \alpha_I) \neq (\alpha'_1, \ldots, \alpha'_I)$, and $F(\alpha'_1, \ldots, \alpha'_I) \geq 0$, we have $F(\alpha_1, \ldots, \alpha_I) = +1$. That is, if x is socially preferred or indifferent to y and some agents raise their consideration of x, then x becomes socially preferred.

Proposition 21.B.1: (May's Theorem) A social welfare functional $F(\alpha_1, \ldots, \alpha_I)$ is a majority voting social welfare functional if and only if it is symmetric among agents, neutral between alternatives, and positive responsive.

Definition 21.C.1: A **social welfare functional** (or **social welfare aggregator**) defined on a given subset $\mathscr{A} \subset \mathscr{R}^I$ is a rule $F : \mathscr{A} \longrightarrow \mathscr{R}$ that assigns a rational preference relation $F(\succsim_1, \ldots, \succsim_I) \in \mathscr{R}$, interpreted as the social preference relation, to any profile of individual rational preference relations $(\succsim_1, \ldots, \succsim_I)$ in the admissible domain $\mathscr{A} \subset \mathscr{R}^I$.

Definition 21.C.2: The social welfare functional $F : \mathscr{A} \longrightarrow \mathscr{R}$ is **Paretian** if, for any pair of alternatives $\{x, y\} \subset X$ and any preference profile $(\succsim_1, \ldots, \succsim_I) \in \mathscr{A}$, we have that x is socially preferred to y, that is, $xF_p(\succsim_1, \ldots, \succsim_I)y$, whenever $x \succ_i y$ for every i.

Definition 21.C.3: The social welfare functional $F : \mathscr{A} \longrightarrow \mathscr{R}$ defined on the domain $\mathscr{A}$ satisfies the **pairwise independence condition** (or **independence of irrelevant alternatives condition**) if the social preference between any two alternatives $\{x, y\} \subset X$ depends only on the profile of individual preferences over the same alternatives. Formally, for any pair of alternatives $\{x, y\} \subset X$, and for any pair of preference profiles $(\succsim_1, \ldots, \succsim_I) \in \mathscr{A}$ and $(\succsim'_1, \ldots, \succsim'_I) \in \mathscr{A}$ with the property that, for every i,

$$x \succsim_i y \Longleftrightarrow x \succsim'_i y \quad \text{ and } \quad y \succsim_i x \Longleftrightarrow y \succsim'_i x,$$

we have that

$$xF(\succsim_1, \ldots, \succsim_I)y \quad \Longleftrightarrow \quad xF(\succsim'_1, \ldots, \succsim'_I)y,$$

and

$$yF(\succsim_1, \ldots, \succsim_I)x \quad \Longleftrightarrow \quad yF(\succsim'_1, \ldots, \succsim'_I)x.$$

Proposition 21.C.1: (Arrow's Impossibility Theorem) Suppose that the number of alternatives is at least three and that the domain of admissible individual profiles, denoted $\mathscr{A}$, is either $\mathscr{A} = \mathscr{R}^I$ or $\mathscr{A} = \mathscr{P}^I$. Then every social welfare functional $F : \mathscr{A} \longrightarrow \mathscr{R}$ that is Paretian and satisfies the pairwise independence condition is **dictatorial** in the following sense: There is an agent h such that, for any $\{x, y\} \subset X$ and any profile $(\succsim_i, \dots, \succsim_I) \in \mathscr{A}$, we have that x is socially preferred to y, that is, $xF_p(\succsim_1, \dots, \succsim_I)y$ whenever $x \succ_h y$.

Definition 21.C.4: Given $F(\cdot)$, we say that a subset of agents $S \subset I$ is :
(i) Decisive for *x* over *y* if whenever every agent in S prefers x to y and every agent not in S prefers y to x, x is socially preferred to y.
(ii) Decisive, if for any pair $x, y \subset X$, S is decisive for x over y.
(iii) Completely decisive for *x* over *y* if whenever every agent in S prefers x to y, x is socially preferred to y.

Definition 21.D.1: Suppose that the preference relation $\succsim$ on X is reflexive and complete. We say then that:
(i) $\succsim$ is **quasitransitive** if the strict preference $\succ$ induced by $\succsim$ (i.e. $x \succ y \iff x \succsim y$ but not $y \succsim x$) is transitive.
(ii) $\succsim$ is **acyclic** if $\succsim$ has a maximal element in every finite subset $X' \subset X$, that is,

$$\{x \in X' : x \succsim y \text{ for all } y \in X'\} \neq \varnothing.$$

Definition 21.D.2: A binary relation $\geq$ on the set of alternatives X is a **linear order** on X if it is **reflexive** (i.e., $x \geq x$ for every $x \in X$), **transitive** (i.e., $x \geq y$ and $y \geq z$ implies $x \geq z$) and **total** (i.e., for any distinct $x, y \in X$, we have either $x \geq y$ or $y \geq x$, but not both).

Definition 21.D.3: The rational preference relation $\succsim$ is **single-peaked** with respect to the linear order $\geq$ on X if there is an alternative $x \in X$ with the property that $\succsim$ is increasing with respect to $\geq$ on $\{y \in X : x \geq y\}$ and decreasing with respect to $\geq$ on $\{y \in X : y \geq x\}$. That is,

$$\text{If } x \geq z > y \quad \text{then} \quad z \succ y$$

and

$$\text{If } y > z \geq x \quad \text{then} \quad z \succ y.$$

Definition 21.D.4: Given a linear order $\geq$ on X, we denote by $\mathscr{R}_{\geq} \subset \mathscr{R}$ the collection of all rational preference relations that are single peaked with respect to $\geq$.

Definition 21.D.5: Agent $h \in I$ is a **median agent for the profile** $(\succsim_1, \dots, \succsim_I) \in \mathscr{R}_{\geq}^I$ if

$$\#\{i \in I : x_i \geq x_h\} \geq \frac{I}{2} \qquad \text{and} \qquad \#\{i \in I : x_h \geq x_i\} \geq \frac{I}{2}.$$

Proposition 21.D.1: Suppose that $\geq$ is a linear order on X and consider a profile of preferences $(\succsim_1, \ldots, \succsim_I)$ where, for every i, $\succsim_i$ is single peaked with respect to $\geq$. Let $h \in I$ be a median agent. Then $x_h \hat{F}(\succsim_1, \ldots, \succsim_I) y$ for every $y \in X$. That is, the peak x_h of the median agent cannot be defeated by majority voting by any other alternative. Any alternative having this property is called a **Condorcet winner**. Therefore, a Condorcet winner exists whenever the preferences of all agents are single-peaked with respect to the same linear order.

Proposition 21.D.2: Suppose that I is odd and that $\geq$ is a linear order on X. Then pairwise majority voting generates a well-defined social welfare functional $F : \mathscr{P}^I_{\geq} \longrightarrow \mathscr{R}$. That is, on the domain of preferences that are single-peaked with respect to $\geq$ and, moreover, have the property that no two distinct alternatives are indifferent, we can conclude that the social relation $\hat{F}(\succsim_1, \ldots, \succsim_I)$ generated by pairwise majority voting is complete and transitive.

Definition 21.E.1: Given any subset $\mathscr{A} \subset \mathscr{R}^I$, a **social choice function** $f : \mathscr{A} \longrightarrow X$ defined on $\mathscr{A}$ assigns a chosen element $f(\succsim, \ldots, \succsim_I) \in X$ to every profile of individual preferences in $\mathscr{A}$.

Definition 21.E.2: The social choice function $f : \mathscr{A} \longrightarrow X$ defined on $\mathscr{A} \subset \mathscr{R}^I$ is **weakly Paretian** if for any profile $(\succsim_1, \ldots, \succsim_I) \in \mathscr{A}$ the choice $f(\succsim_1, \ldots, \succsim_I) \in X$ is a weak Pareto optimum. That is, if for some pair $\{x, y\} \subset X$ we have that $x \succ_i y$ for every i, then $y \neq f(\succsim_1, \ldots, \succsim_I)$.

Definition 21.E.3: The alternative $x \in X$ **maintains its position** from the profile $(\succsim_1, \ldots, \succsim_I) \in \mathscr{R}^I$ to the profile $(\succsim'_1, \ldots, \succsim'_I) \in \mathscr{R}^I$ if

$$x \succsim_i y \text{ implies } x \succsim'_i y$$

for every i and every $y \in X$.

Definition 21.E.4: The social choice function $f : \mathscr{A} \longrightarrow X$ defined on $\mathscr{A} \subset \mathscr{R}^I$ is **monotonic** if for any two profiles $(\succsim_1, \ldots, \succsim_I) \in \mathscr{A}$, $(\succsim'_1, \ldots, \succsim'_I) \in \mathscr{A}$ with the property that the chosen alternative $x = f(\succsim_1, \ldots, \succsim_I)$ maintains its position from $(\succsim_1, \ldots, \succsim_I)$ to $(\succsim'_1, \ldots, \succsim'_I)$, we have thats $f(\succsim'_1, \ldots, \succsim'_I) = x$.

Definition 21.E.5: An agent $h \in I$ is a **dictator** for the social choice function $f : \mathscr{A} \longrightarrow X$ if, for every profile $(\succsim_1, \ldots, \succsim_I) \in \mathscr{A}$, $f(\succsim_1, \ldots, \succsim_I)$ is a most preferred alternative for $\succsim_h$ in X; that is,

$$f(\succsim_1, \ldots, \succsim_I) \in \{x \in X : x \succsim_h y \text{ for every } y \in X\}.$$

A social choice function that admits a dictator is called **dictatorial**.

Proposition 21.E.1: Suppose that the number of alternatives is at least three and that the domain of admissible preference profiles is either $\mathscr{A} = \mathscr{R}^I$ or $\mathscr{A} = \mathscr{P}^I$. Then every weakly Paretian and monotonic social choice function $f : \mathscr{A} \longrightarrow X$ is dictatorial.

Definition 21.E.6: Given a finite subset $X' \subset X$ and a profile $(\succsim_1, \dots, \succsim_I) \in \mathscr{R}^I$, we say that the profile $(\succsim'_1, \dots, \succsim'_I)$ takes X' to the top from $(\succsim_1, \dots, \succsim_I)$ if, for every i,

$$\begin{array}{ll} x \succ'_i y & \text{for } x \in X' \text{ and } y \notin X', \\ x \succsim'_i y \Longleftrightarrow x \succsim'_i y & \text{for all } x, y \in X'. \end{array}$$

Proposition 21.E.2: Suppose that the number of alternatives is at least three and that $f : \mathscr{P}^I \longrightarrow X$ is a social choice function that is weakly Paretian and satisfies the following **no-incentive-to-misrepresent** condition:

$$f(\succsim_1, \dots, \succsim_{h-1}, \succsim_h, \succsim_{h+1}, \dots, \succsim_I) \succsim_h f(\succsim_1, \dots, \succsim_{h-1}, \succsim'_h, \succsim_{h+1}, \dots, \succsim_I)$$

for every agent h, every $\succsim'_h \in \mathscr{P}$, and profile $(\succsim_1, \dots, \succsim_I) \in \mathscr{P}^I$. Then $f(\cdot)$ is dictatorial.

CHAPTER 22. ELEMENTS OF WELFARE ECONOMICS AND AXIOMATIC BARGAINING

Definition 22.B.1: The **utility possibility set (UPS)** is the set

$$U = \{(u_1, \dots, u_I) \in \mathbb{R}^I : u_1 \leq u_i(x), \dots, u_I \leq u_I(x) \text{ for some } x \in X\} \subset \mathbb{R}^L.$$

The **Pareto frontier** of U is formed by the utility vectors $u = (u_1, \dots, u_I) \in U$ for which there is no other $u' = (u'_1, \dots, u'_I) \in U$ with $u_i \geq u_i$ for every i and $u'_i > u_i$ for some i.

Definition 22.D.1: Given a set X of alternatives, a **social welfare functional** $F : \mathscr{U}^I \longrightarrow \mathscr{R}$ is a rule that assigns a rational preference relation $F(\tilde{u}_1, \dots, \tilde{u}_I)$ among the alternatives in the domain X to every possible profile of individual utility functions $(\tilde{u}_1(\cdot), \dots, \tilde{u}_I(\cdot))$ defined on X. The strict preference relation derived from $F(\tilde{u}_1, \dots, \tilde{u}_I)$ is denoted $F_p(\tilde{u}_1, \dots, \tilde{u}_I)$.

Definition 22.D.2: The social welfare functional $F : \mathscr{U}^I \longrightarrow \mathscr{R}$ satisfies the (weak) **Pareto property**, or is **Paretian**, if, for any profile $(\tilde{u}_1, \dots, \tilde{u}_I) \in \mathscr{U}^I$ and any pair $x, y \in X$, we have that $\tilde{u}_i(x) \geq \tilde{u}_i(y)$ for all i implies $xF(\tilde{u}_1, \dots, \tilde{u}_I)y$, and also that $\tilde{u}_i(x) > \tilde{u}_i(y)$ for all i implies $xF_p(\tilde{u}_1, \dots, \tilde{u}_I)y$.

Definition 22.D.3: The social welfare functional $F : \mathscr{U}^I \longrightarrow \mathscr{R}$ satisfies the **pairwise independence condition** if, whenever $x, y \in X$ are two alternatives and $(\tilde{u}_1, \dots, \tilde{u}_I) \in \mathscr{U}^I$, $(\tilde{u}'_1, \dots, \tilde{u}'_I) \in \mathscr{U}^I$ are two utility function profiles with $\tilde{u}_i(x) = \tilde{u}'_i(x)$ and $\tilde{u}_i(y) = \tilde{u}'_i(y)$ for all i, we have

$$xF(\tilde{u}_1, \dots, \tilde{u}_I)y \quad \Longleftrightarrow \quad xF(\tilde{u}'_1, \dots, \tilde{u}'_I)y.$$

Proposition 22.D.1: Suppose that there are at least three alternatives in X and that the Paretian social welfare functional $F : \mathscr{U}^I \longrightarrow \mathscr{R}$ satisfies the pairwise independence condition. Then there is a rational preference relation $\succsim$ defined on $\mathbb{R}^I$ [that is, on profiles $(u_1, \ldots, u_I) \in \mathbb{R}^I$ of individual utility values] that generates $F(\cdot)$. In other words, for every profile of utility functions $(\tilde{u}_1, \ldots, \tilde{u}_I) \in \mathscr{U}^I$ and for every pair of alternatives $x, y \in X$ we have

$$xF(\tilde{u}_1, \ldots, \tilde{u}_I)y \quad \Longleftrightarrow \quad \big(\tilde{u}_1(x), \ldots, \tilde{u}_I(x)\big) \succsim \big(\tilde{u}_1(y), \ldots, \tilde{u}_I(y)\big).$$

Definition 22.D.4: We say that the social welfare functional $F : \mathscr{U}^I \longrightarrow \mathscr{R}$ is **invariant to common cardinal transformations** if $F(\tilde{u}_1, \ldots, \tilde{u}_I) = F(\tilde{u}'_1, \ldots, \tilde{u}'_I)$ whenever the profiles of utility functions $(\tilde{u}_1, \ldots, \tilde{u}_I)$ and $(\tilde{u}'_1, \ldots, \tilde{u}'_I)$ differ only by a common change of origin and units, that is, whenever there are numbers $\beta > 0$ and α such that $\tilde{u}_i(x) = \beta \tilde{u}'_i(x) + \alpha$ for all i and $x \in X$. If the invariance is only with respect to common changes of origin (i.e., we require $\beta = 1$) or of units (i.e., we require $\alpha = 0$), then we say that $F(\cdot)$ is **invariant to common changes of origin** or **of units**, respectively.

Proposition 22.D.2: Suppose that the social welfare functional $F : \mathscr{U}^I \longrightarrow \mathscr{R}$ is generated from a continuous and increasing social welfare function. Suppose also that $F(\cdot)$ is invariant to common changes of origins. Then the social welfare functional can be generated from a social welfare function of the form

$$W(u_1, \ldots, u_I) = \bar{u} - g(u_1 - \bar{u}, \ldots, u_I - \bar{u}),$$

where $\bar{u} = \frac{1}{I}\sum_i u_i$.

Moreover, if $F(\cdot)$ is also independent of common changes of units, that is, fully invariant to common cardinal transformations, then $g(\cdot)$ is homogeneous of degree one on its domain: $\{s \in \mathbb{R}^I : \sum_i s_i = 0\}$.

Definition 22.D.5: The social welfare functional $F : \mathscr{U}^I \longrightarrow \mathscr{R}$ does not allow **interpersonal comparisons of utility** $F(\tilde{u}_1, \ldots, \tilde{u}_I) = F(\tilde{u}'_1, \ldots, \tilde{u}'_I)$ whenever there are numbers $\beta_i > 0$ and α_i such that $\tilde{u}_i(x) = \beta \tilde{u}'_i(x) + \alpha_i$ for all i and x. If the invariance is only with respect to independent changes of origin (i.e., we require $\beta_i = 1$ for all i), or only with respect to independent changes of units (i.e., we require that $\alpha_i = 0$ for all i), then we say that $F(\cdot)$ is **invariant to independent changes of origins** or **of units**, respectively.

Proposition 22.D.3: Suppose that the social welfare functional $F : \mathscr{U}^I \longrightarrow \mathscr{R}$ can be generated from an increasing, continuous social welfare function. If $F(\cdot)$ is invariant to independent changes of origins, then $F(\cdot)$ can be generated from a social welfare function $W(\cdot)$ of the purely utilitarian (but possibly nonsymmetric) form. That is, there are constants $b_i \geq 0$, not all zero, such that

$$W(u_1, \dots, u_I) = \sum_i b_i u_i \qquad \text{for all } i.$$

Moreover, if $F(\cdot)$ is also invariant to independent changes of units [i.e., if $F(\cdot)$ does not allow for interpersonal comparisons of utility], then F is dictatorial: There is an agent h such that, for every pair $x, y \in X$, $\tilde{u}_h(x) > \tilde{u}_h(y)$ implies $x F_p(\tilde{u}_1, \dots, \tilde{u}_I) y$.

Definition 22.E.1: A **bargaining solution** is a rule that assigns a solution vector $f(U, u^*) \in U$ to every bargaining problem (U, u^*).

Definition 22.E.2: The bargaining solution $f(\cdot)$ is **independent of utility origins (IUO)** or **invariant to independent changes of origins**, if for any $\alpha = (\alpha_1, \dots, \alpha_I) \in \mathbb{R}^I$ we have

$$f_i(U', u^* + \alpha) = f_i(U, u^*) + \alpha_i \qquad \text{for every } i$$

whenever $U' = \{(u_1 + \alpha_1, \dots, u_I + \alpha_I) : u \in U\}$.

Definition 22.E.3: The bargaining solution $f(\cdot)$ is **independent of utility units (IUU)**, or **invariant to independent changes of units**, if for any $\beta = (\beta_1, \dots, \beta_I) \in \mathbb{R}^I$ with $\beta_i > 0$ for all i, we have

$$f_i(U') = \beta_{f\,i}(U) \quad \text{for every } i$$

whenever $U' = \{(\beta u_1, \dots, \beta u_I) : u \in U\}$

Definition 22.E.4: The bargaining solution $f(\cdot)$ satisfies the **Pareto property (P)**, or is **Paretian**, if, for every U, $f(U)$ is a (weak) Pareto optimum, that is, there is no $u \in U$ such that $u_i > f_i(U)$ for every i.

Definition 22.E.5: The bargaining solution $f(\cdot)$ satisfies the property of **symmetry (S)** if whenever $U \subset \mathbb{R}^I$ is a symmetric set (i.e., U remains unaltered under permutations of the axes), we have that all the entries of $f(U)$ are equal.

Definition 22.E.6: The bargaining solution $f(\cdot)$ satisfies the property of **individual rationality (IR)** if $f(U) \geq 0$.

Definition 22.E.7: The bargaining solution satisfies the property of **independence of irrelevant alternatives (IIA)** if, whenever $U' \subset U$ and $f(U) \in U'$, it follows that $f(U') = f(U)$.

Proposition 22.E.1: The Nash solution is the only bargaining solution that is independent of utility origins and units, Paretian, symmetric, and independent of irrelevant alternatives.

Definition 22.F.1: Given the set of agents I, a **cooperative solution** $f(\cdot)$ is a rule that assigns to every game $v(\cdot)$ in characteristic form a utility allocation $f(v) \in \mathbb{R}^I$ that is feasible for the entire group, that is, such that $\sum_i f_i(v) \leq v(I)$.

Definition 22.F.2: The cooperative solution $f(\cdot)$ is **independent of utility origins and of common changes of utility units** if, whenever we have two characteristic forms $v(\cdot)$ and $v'(\cdot)$ such that $v(S) = \beta v'(S) + \sum_{i \in S} \alpha_i$ for every $S \subset I$ and some numbers $\alpha_1, \ldots, \alpha_I$, and $\beta > 0$, it follows that $f(v) = \beta f(v') + (\alpha_1, \ldots, \alpha_I)$.

Definition 22.F.3: The cooperative solution $f(\cdot)$ is **Paretian** if $\sum_i f_i(v) = v(I)$ for every characteristic form $v(\cdot)$.

Definition 22.F.4: The cooperative solution $f(\cdot)$ is **symmetric** if the following property holds: Suppose that two characteristic forms, $v(\cdot)$ and $v'(\cdot)$ differ only by a permutation $\pi : I \longrightarrow I$ of the names of the agents; that is, $v'(S) = v(\pi(S))$ for all $s \subset I$. Then the solution also differs only by this permutation; that is, $f_i(v') = f_{\pi(i)}(v)$ for all i.

Definition 22.F.5: The cooperative solution $f(\cdot)$ satisfies the **dummy axiom** if, for all games $v(\cdot)$ and all agents i such that $v(S \cup \{i\}) = v(S)$ for all $S \subset I$, we have $f_i(v) = v(i)$ $(= 0)$. In words: If agent i is **dummy** (i.e., does not contribute anything to any coalition), then agent i does not receive any share of the surplus.

Definition 22.F.6: The **Shapely value** solution $f_s(\cdot)$ is defined by

$$f_{si}(v) = \frac{1}{I!} \sum_{\pi} g_{v,\pi}(i) \qquad \text{for every } i.$$

CHAPTER 23. THE MECHANISM DESIGN PROBLEM

Definition 23.B.1: A **social choice function** is a function $f : \Theta_1 \times \cdots \times \Theta_I \longrightarrow X$ that, for each possible profile of the agents' types $(\theta_1, \ldots, \theta_I)$, assigns a collective choice $f(\theta_1, \ldots, \theta_I) \in X$.

Definition 23.B.2: The social choice function $f : \Theta_1 \times \cdots \times \Theta_I \longrightarrow X$ is **ex post efficient** (or **Paretian)** if for no profile $\theta = (\theta_1, \ldots, \theta_I)$ is there an $x \in X$ such that $u_i(x, \theta_i) \geq u_i(f(\theta), \theta_i)$ for every i, and $u_i(x, \theta_i) > u_i(f(\theta), \theta_i)$ for some i.

Definition 23.B.3: A **mechanism** $\Gamma = (S_1, \ldots, S_I, g(\cdot))$ is a collection of I strategy sets $(S_1, \ldots, S_I)$ and an outcome function $g : S_1 \times \cdots \times S_I \longrightarrow X$.

Definition 23.B.4: The mechanism $\Gamma = (S_1, \ldots, S_I, g(\cdot))$ **implements** social choice function $f(\cdot)$ if there is an equilibrium strategy profile $(s_1^*(\cdot), \ldots, s_I^*(\cdot))$ of the game induced by Γ such that $g(s_1^*(\theta_1), \ldots, s_I^*(\theta_I)) = f(\theta_1, \ldots, \theta_I)$ for all $(\theta_1, \ldots, \theta_I) \in \Theta_1 \times \cdots \times \Theta_I$.

Definition 23.B.5: A **direct revelation mechanism** is a mechanism in which $S_i = \Theta_i$ for all i and $g(\theta) = f(\theta)$ for all $\theta \in \Theta_1 \times \cdots \times \Theta_I$.

Definition 23.B.6: The social choice function $f(\cdot)$ is **truthfully implementable** (or **incentive compatible**) if the direct revelation mechanism $\Gamma = (\Theta_1, \ldots, \Theta_I, f(\cdot))$ has an equilibrium $(s_1^*(\cdot), \ldots, s_I^*(\cdot))$ in which $s_i^*(\theta_i) = \theta_i$ for all $\theta_i \in \Theta_i$ and all $i = 1, \ldots, I$; that is, if truth telling by each agent i constitutes an equilibrium of $\Gamma = (\Theta_1, \ldots, \Theta_I, f(\cdot))$.

Definition 23.C.1: The strategy profile $s^*(\cdot) = (s_1^*(\cdot), \ldots, s_I^*(\cdot))$ is a **dominant strategy equilibrium** of mechanism $\Gamma = (S_1, \ldots, S_I, g(\cdot))$ if, for all i and all $\theta_i \in \Theta_i$,

$$u_i\Big(g\big(s_i^*(\theta_i), s_{-i}\big), \theta_i\Big) \geq u_i\Big(g\big(s_i', s_{-i}\big), \theta_i\Big)$$

for all $s_i' \in S_i$ and all $s_{-i} \in S_{-i}$.

Definition 23.C.2: The mechanism $\Gamma = (S_1, \ldots, S_I, g(\cdot))$ **implements the social choice function** $f(\cdot)$ **in dominant strategies** if there exists a dominant strategy equilibrium of Γ, $s^*(\cdot) = \big(s_1^*(\cdot), \ldots, s_I^*(\cdot)\big)$, such that $g\big(s^*(\theta)\big) = f(\theta)$ for all $\theta \in \Theta$.

Definition 23.C.3: The social choice function $f(\cdot)$ is **truthfully implementable in dominant strategies** (or **dominant strategy incentive compatible**, or **strategy-proof**, or **straightforward**) if $s_i^*(\theta_i) = \theta_i$ for all $\theta_i \in \Theta_i$ and $i = 1, \ldots, I$ is a dominant strategy equilibrium of the direct revelation mechanism $\Gamma = (\Theta_1, \ldots, \Theta_I, f(\cdot))$. That is, if for all i and all $\theta_i \in \Theta_i$,

$$u_i\Big(f\big(\theta_i, \theta_{-i}\big), \theta_i\Big) \geq u_i\Big(f\big(\hat{\theta}_i, \theta_{-i}\big), \theta_i\Big)$$

for all $\hat{\theta}_i \in \Theta_i$ and all $\theta_{-i} \in \Theta_{-i}$.

Proposition 23.C.1: (The Revelation Principle for Dominant Strategies) Suppose that there exists a mechanism $\Gamma = (S_1, \ldots, S_I, g(\cdot))$ that implements the social choice function $f(\cdot)$ in dominant strategies. Then $f(\cdot)$ is truthfully implementable in dominant strategies.

Proposition 23.C.2: The social choice function $f(\cdot)$ is truthfully implementable in dominant strategies if and only if for all i, all $\theta_{-i} \in \Theta_{-i}$, and all pairs of types for agent i, θ_i' and $\theta_i'' \in \Theta_i$, we have

$$f(\theta_i'', \theta_{-i}) \in L_i\big(f(\theta_i', \theta_{-i}), \theta_i'\big) \quad \text{and} \quad f(\theta_i', \theta_{-i}) \in L_i\big(f(\theta_i'', \theta_{-i}), \theta_i''\big).$$

Definition 23.C.4: The social choice function $f(\cdot)$ is **dictatorial** if there is an agent i such that, for all $\theta = (\theta_1, \ldots, \theta_I) \in \Theta$,

$$f(\theta) \in \{x \in X : u_i(x, \theta_i) \geq u_i(y, \theta_i) \text{ for all } y \in X\}.$$

Definition 23.C.5: The social choice function $f(\cdot)$ is **monotonic** if, for any θ, if θ' is such that $L_i\big(f(\theta), \theta_i\big) \subset L_i\big(f(\theta), \theta_i'\big)$ for all i [i.e., if $L_i\big(f(\theta), \theta_i\big)$ is weakly included in $L_i(f(\theta), \theta_i')$ for all i], then $f(\theta') = f(\theta)$.

Proposition 23.C.3: (The Gibbard-Satterthwaite Theorem) Suppose that X is finite and contains at least three elements, that $\mathscr{R}_i = \mathscr{P}$ for all i, and that $f(\Theta) = X$. Then the social choice function $f(\cdot)$ is truthfully implementable in dominant strategies if and only if it is dictatorial.

Corollary 23.C.1: Suppose that X is finite and contains at least three elements, that $\mathscr{P} \subset \mathscr{R}_i$ for all i, and that $f(\Theta) = X$. Then the social choice function $f(\cdot)$ is truthfully implementable in dominant strategies if and only if it is dictatorial.

Definition 23.C.6: The social choice function $f(\cdot)$ is **dictatorial on set** $\widehat{X} \subset X$ if there exists an agent i such that, for all $\theta = (\theta_1, \ldots, \theta_I) \in \Theta$,

$$f(\theta) \in \{x \in X : u_i(x, \theta_i) \geq u_i(y, \theta_i) \text{ for all } y \in \widehat{X}\}\ .$$

Corollary 23.C.2: Suppose that X is finite, that the number of elements in $f(\Theta)$ is at least three, and that $\mathscr{P} \subset \mathscr{R}_i$ for all $i = 1, \ldots, I$. Then $f(\cdot)$ is truthfully implementable in dominant strategies if and only if it is dictatorial on the set $f(\Theta)$.

Proposition 23.C.4: Let $k^*(\cdot)$ be a function satisfying

$$\sum_{i=1}^{I} v_i\big(k(\theta), \theta_i\big) \geq \sum_{i=1}^{I} v_i(k, \theta_i) \quad \text{for all} \quad k \in K.$$

The social choice function $f(\cdot) = \big(k^*(\cdot), t_1(\cdot), \ldots, t_I(\cdot)\big)$ is truthfully implementable in dominant strategies if, for all $i = 1, \ldots, I$,

$$t_i(\theta) = \left[\sum_{j \neq i} v_j\big(k^*(\theta), \theta_j\big)\right] + h_i(\theta_{-i}),$$

where $h_i(\cdot)$ is an arbitrary function of θ_{-i}.

Proposition 23.C.5: Suppose that for each agent $i = 1, \ldots, I$, $\{v_i(\cdot, \theta_i) : \theta_i \in \Theta_i\} = \mathscr{V}$; that is, every possible valuation function from K to $\mathbb{R}$ arises for some $\theta_i \in \Theta_i$. Then a social choice function $f(\cdot) = \big(k^*(\cdot), t_1(\cdot), \ldots, t_I(\cdot)\big)$ in which $k^*(\cdot)$ satisfies $\sum_{i=1}^I v_i\big(k(\theta), \theta_i\big) \geq \sum_{i=1}^I v_i(k, \theta_i)$ for all $k \in K$ is truthfully implementable in dominant strategies only if $t_i(\cdot)$ satisfies

$$t_i(\theta) = \left[\sum_{j \neq i} v_j\big(k^*(\theta), \theta_j\big)\right] + h_i(\theta_{-i}),$$

for all $i = 1, \ldots, I$.

Proposition 23.C.6: Suppose that for each agent $i = 1, \ldots, I$, $\{v_i(\cdot, \theta_i) : \theta_i \in \Theta_i\} = \mathscr{V}$; that is, every possible valuation function from K to $\mathbb{R}$ arises for some $\theta_i \in \Theta_i$. Then there is no social choice function $f(\cdot) = \big(k^*(\cdot), t_1(\cdot), \ldots, t_I(\cdot)\big)$ that is truthfully implementable in dominant strategies and is ex post efficient, that is, that satisfies

$$\sum_{i=1}^I v_i\big(k(\theta), \theta_i\big) \geq \sum_{i=1}^I v_i(k, \theta_i) \quad \text{for all } \ k \in K,$$

and the **budget balance condition**,

$$\sum_i t_i(\theta) = 0 \text{ for all } \theta \in \Theta.$$

Definition 23.D.1: The strategy profile $s^*(\cdot) = \big(s_1^*(\cdot), \ldots, s_I^*(\cdot)\big)$ is a **Bayesian Nash equilibrium** of mechanism $\Gamma = \big(S_1, \ldots, S_I, g(\cdot)\big)$, if, for all i and all $\theta_i \in \Theta_i$,

$$\mathbb{E}_{\theta-i}\left[u_i\Big(g\Big(s_i^*(\theta_i), s_{-i}^*(\theta_{-i})\Big), \theta_i\Big) \;\Big|\; \theta_i\right] \geq \mathbb{E}_{\theta-i}\left[u_i\Big(g\Big(\hat{s}_i, s_{-i}^*(\theta_{-i})\Big), \theta_i\Big) \;\Big|\; \theta_i\right]$$

for all $\hat{s}_i \in S_i$.

Definition 23.D.2: The mechanism $\Gamma = \big(S_1, \ldots, S_I, g(\cdot)\big)$ **implements the social choice function** $f(\cdot)$ **in Bayesian Nash Equilibrium** if there is a Bayesian Nash equilibrium of Γ, $s^*(\cdot) = \big(s_1^*(\cdot), \ldots, s_I^*(\cdot)\big)$ such that $g\big(s^*(\theta)\big) = f(\theta)$ for all $\theta \in \Theta$.

Definition 23.D.3: The social choice function $f(\cdot)$ is **truthfully implementable in Bayesian Nash equilibrium** (or **Bayesian incentive compatible**) if $s_i^*(\theta_i) = \theta_i$ for all $\theta_i \in \Theta_i$ and $i = 1, \ldots, I$ is a Bayesian Nash equilibrium of the direct revelation mechanism $\Gamma = \big(\Theta_1, \ldots, \Theta_I, f(\cdot)\big)$. That is, if for all $i = 1, \ldots, I$ and all $\theta_i \in \Theta_i$,

$$\mathbb{E}_{\theta-i}\left[u_i\Big(f(\theta_i, \theta_{-i}), \theta_i\Big) \;\Big|\; \theta_i\right] \geq \mathbb{E}_{\theta-i}\left[u_i\Big(f(\hat{\theta}_i, \theta_{-i}), \theta_i\Big) \;\Big|\; \theta_i\right]$$

for all $\hat{\theta}_i \in \Theta_i$.

Proposition 23.D.1: (The Revelation Principle for Bayesian Nash Equilibrium) Suppose that there exists a mechanism $\Gamma = (S_1, \ldots, S_I, g(\cdot))$ that implements the social choice function $f(\cdot)$

in Bayesian Nash equilibrium. Then $f(\cdot)$ is truthfully implementable in Bayesian Nash equilibrium.

Proposition 23.D.2: The social choice function $f(\cdot) = \big(k(\cdot), t_1(\cdot), \ldots, t_I(\cdot)\big)$ is a Bayesian incentive compatible if and only if, for all $i = 1, \ldots, I$,
(i) $\bar{v}_i(\cdot)$ is nondecreasing.
(ii) $U_i(\theta_i) = U_i(\underline{\theta}_i) + \displaystyle\int_{\underline{\theta}_i}^{\theta_i} \bar{v}_i(s)ds \qquad$ for all θ_i.

Proposition 23.D.3: (The Revenue Equivalence Theorem) Consider an auction setting with I risk-neutral buyers, in which buyer i's valuation is drawn from an interval $\big[\underline{\theta}_i, \bar{\theta}_i\big]$ with $\underline{\theta}_i \neq \bar{\theta}_i$ and a strictly positive density $\phi_i(\cdot) > 0$, and in which buyers' types are statistically independent. Suppose that a given pair of Bayesian Nash equilibria of two different auction procedures are such that for every buyer i: **(i)** For each possible realization of $(\theta_1, \ldots, \theta_I)$, the buyer i has an identical probability of getting the good in the two auctions; and **(ii)** Buyer i has the same expected utility level in the two auctions when his valuation for the object is at its lowest possible level. Then these equilibria of the two auctions generate the same expected revenue for the seller.

Definition: Consider a social choice function $f(\cdot) = \big(k(\cdot), t_1(\cdot), \ldots, t_I(\cdot)\big)$. For all $i = 1, \ldots, I$, the **expected benefits** of agents $j \neq i$ is

$$\xi_i(\theta_i) = \mathbb{E}_{\tilde{\theta}_{-i}}\left[\sum_{j \neq i} v_j\big(k^*(\theta_i, \tilde{\theta}_{-i}), \tilde{\theta}_j\big)\right],$$

when agent i announces his type to be θ_i and agents $j \neq i$ tell the truth. Let the transfers $t_i(\cdot)$ satisfy

$$t_i(\theta) = \xi_i(\theta_i) + h_i(\theta_{-i}),$$

for all $i = 1, \ldots, I$, for some arbitrary function $h_i(\theta_{-i})$. The **expected externality** is the change in this transfer when agent i changes his announced type. The **expected externality mechanism** is the direct revelation mechanism by letting

$$h_i(\theta_{-i}) = \frac{1}{I-1}\sum_{j \neq i} \xi_j(\theta_j).$$

Proposition 23.E.1: (The Myerson-Satterthwaite Theorem) Consider a bilaeral trade setting in which the buyer and seller are risk neutral, the valuations θ_1 and θ_2 are independently drawn from the intervals $[\underline{\theta}_1, \bar{\theta}_1] \subset \mathbb{R}$ and $[\underline{\theta}_2, \bar{\theta}_2] \subset \mathbb{R}$ with strictly positive densities, and $(\underline{\theta}_1, \bar{\theta}_1) \cap (\underline{\theta}_2, \bar{\theta}_2) \neq \varnothing$. Then there is no Bayesian incentive compatible social choice function that is ex post efficient and gives every buyer type and every seller type nonnegative expected gains from participation.

Definition 23.F.1: Given any set of feasible social choice functions F, the social choice function $f(\cdot) \in F$ is **ex ante efficient in F** if there is no $\hat{f}(\cdot) \in F$ having the property that $U_i(\hat{f}) \geq U_i(f)$ for all $i = 1, \dots, I$, and $U_i(\hat{f}) > U_i(f)$ for some i.

Definition 23.F.2: Given any set of feasible social choice functions F, the social choice function f$(\cdot) \in F$ is **interim efficient in F** if there is no $\hat{f}(\cdot) \in F$ having the property that $U_i(\theta_i | \hat{f}) \geq U_i(\theta_i | f)$ for all $\theta_i \in \Theta_i$ and all $i = 1, \dots, I$, and $U_i(\theta_i | \hat{f}) > U_i(\theta_i | f)$ for some i and $\theta_i \in \Theta_i$.

Proposition 23.F.1: Given any set of feasible social choice functions F, if the social choice function $f(\cdot) \in F$ is ex ante efficient in F, then it is also interim efficient in F.

Definition 23.F.3: Given any set of feasible social choice functions F, the social choice function $f(\cdot) \in F$ is **ex post efficient in *F*** if there is no $\hat{f}(\cdot) \in F$ having the property that $u_i\big(\hat{f}(\theta), \theta_i\big) \geq u_i\big(f(\theta), \theta_i\big)$ for all $i = 1, \dots, I$ and all $\theta \in \Theta$, and $u_i\big(\hat{f}(\theta), \theta_i\big) > u_i\big(f(\theta), \theta_i\big)$ for some i and $\theta \in \Theta$.

Definition 23.AA.1: The mechanism $\Gamma = (S_1, \dots, S_I, g(\cdot))$ **strongly implements** social choice function $f : \Theta_1 \times \cdots \times \Theta_I \longrightarrow X$ if every equilibrium strategy profile $\big(s_1^*(\cdot), \dots, s_I^*(\cdot)\big)$ of the game induced by Γ has the property that $g\big(s_1^*(\cdot), \dots, s_I^*(\cdot)\big) = f(\theta_1, \dots, \theta_I)$ for all $(\theta_1, \dots, \theta_I)$.

Definition 23.BB.1: The mechanism $\Gamma = \big(S_1, \dots, S_I, g(\cdot)\big)$ **implements the social choice function** $f(\cdot)$ **in Nash equilibrium** if, for each profile of the agents' preference parameters $\theta = (\theta_1, \dots, \theta_I) \in \Theta$, there is a Nash equilibrium of the game induced by Γ, $s^*(\theta) = \big(s_1^*(\cdot), \dots, s_I^*(\cdot)\big)$, such that $g\big(s^*(\theta)\big) = f(\theta)$. The mechanism $\Gamma = (S_1, \dots, S_I, g(\cdot))$ **strongly implements the social choice function** $f(\cdot)$ **in Nash equilibrium** if, for each profile of the agents' preference parameters $\theta = (\theta_1, \dots, \theta_I) \in \Theta$, every Nash equilibrium of the game induced by Γ results in outcome $f(\theta)$.

Proposition 23.BB.1: If the social choice function $f(\cdot)$ can be strongly implemented in Nash equilibrium, then $f(\cdot)$ is monotonic.

Proposition 23.BB.2: Suppose that X is finite and contains at least three elements, that $\mathscr{R}_i = \mathscr{P}$ for all i, and that $f(\Theta) = X$. Then the social choice function $f(\cdot)$ is strongly implementable in Nash equilibrium if and only if it is dictatorial.

Proposition 23.BB.3: If $I \geq 3$, $f(\cdot)$ is monotonic, and $f(\cdot)$ satisfies no veto power, then $f(\cdot)$ is strongly implementable in Nash equilibrium.