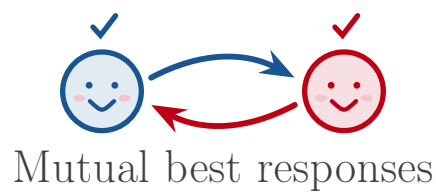


Chapter 9

Nash Equilibrium



Beliefs and Actual Choices

Rationalizability allows a player to choose a best response to a belief that is wrong about what the other player actually does.

If each player's belief correctly predicts the others' choices, rationality requires every choice to be a best response to the actual opposing choices. The resulting profile consists of **mutual best responses**.

Agreement, shared conventions (e.g., walking on the right side of a hallway), or experience may help align beliefs and choices. Rationality alone does not ensure that alignment.

Nash Equilibrium

Definition. A **pure-strategy Nash equilibrium (NE)** is a profile $s^* = (s_1^*, \dots, s_n^*)$ in which every player's strategy is a best response to the others' strategies:

$$s_i^* \in BR_i(s_{-i}^*) \quad \text{for every player } i.$$

Equivalently, for each player i and every alternative $s_i \in S_i$,

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*).$$

No player can gain by changing her own strategy while the others' strategies stay fixed. Equilibrium is a property of a whole profile.

Discussion: The Stag Hunt

	Stag	Hare
Stag	5, 5	0, 4
Hare	4, 0	4, 4

Find every pure-strategy Nash equilibrium. For each remaining profile, identify a player who can gain by changing her own action.

All four profiles are rationalizable. Does this make all four equilibria?

Answer: Mutual Best Responses

	Stag	Hare
Stag	5, 5	0, 4
Hare	4, 0	4, 4

Both (Stag, Stag) and (Hare, Hare) are equilibria. Against Stag, Stag gives $5 > 4$; against Hare, Hare gives $4 > 0$.

At either mismatch, the stag hunter can switch to Hare and raise her payoff from 0 to 4. Rationalizability can allow such a mismatch; Nash equilibrium cannot.

Discussion: The Prisoners' Dilemma

	C	D
C	2, 2	0, 3
D	3, 0	1, 1

Here C means cooperate and D means defect. Find every pure-strategy Nash equilibrium.

Both players gain if they move together from (D, D) to (C, C) . Does this show that (D, D) is not an equilibrium?

Answer: Unilateral and Joint Changes

	C	D
C	2, 2	0, 3
D	3, 0	1, 1

D is each player's unique best response to either action, so (D, D) is the unique Nash equilibrium.

A unilateral switch to C at (D, D) reduces the deviator's payoff from 1 to 0. The gain from a joint switch does not violate the equilibrium condition, which holds the other player's action fixed.

Finding Equilibria in a Matrix

Player 1 chooses a row and player 2 a column; payoffs are listed in that order. The procedure below is informally called the “underlining method” or “circling method.”

1. In each column, mark every largest payoff of player 1. These are player 1's best responses to that column.
2. In each row, mark every largest payoff of player 2. These are player 2's best responses to that row.
3. A cell is a Nash equilibrium exactly when both payoffs are marked.

Mark all ties. Different best responses can be optimal against the same opponent strategy.

Discussion: Find All Equilibria

	X	Y	Z
J	5, 6	3, 7	0, 4
K	8, 3	3, 1	5, 2
L	7, 5	4, 4	5, 6
M	3, 5	7, 5	3, 3

Find both players' best responses and all pure-strategy Nash equilibria. Are any best responses tied?

Answer: Player 1's Best Responses

	X	Y	Z
J	5, 6	3, 7	0, 4
K	<u>8</u> , 3	3, 1	<u>5</u> , 2
L	7, 5	4, 4	<u>5</u> , 6
M	3, 5	<u>7</u> , 5	3, 3

$$BR_1(X) = \{K\}, \quad BR_1(Y) = \{M\}, \quad BR_1(Z) = \{K, L\}.$$

Both K and L give player 1 a payoff of 5 against Z .

Answer: Three Equilibrium Profiles

	X	Y	Z
J	5, 6	3, <u>7</u>	0, 4
K	<u>8</u> , <u>3</u>	3, 1	<u>5</u> , 2
L	7, 5	4, 4	<u>5</u> , <u>6</u>
M	3, <u>5</u>	<u>7</u> , <u>5</u>	3, 3

Player 2's best responses to J, K, L, M are, respectively, $\{Y\}$, $\{X\}$, $\{Z\}$, and $\{X, Y\}$.

The equilibrium profiles are (K, X) , (L, Z) , and (M, Y) .

Strict Nash Equilibrium

Definition. A Nash equilibrium is **strict** if each player's equilibrium strategy is her unique best response to the others' strategies.

Every unilateral change to a different strategy then lowers the deviator's payoff strictly.

In the 4×3 game, (K, X) is strict. At (L, Z) , player 1 can switch to K without losing; at (M, Y) , player 2 can switch to X without losing. Those two equilibria are not strict.

Discussion: Matching Pennies

	H	T
H	$1, -1$	$-1, 1$
T	$-1, 1$	$1, -1$

Player 1 wants the coins to match; player 2 wants them to differ. Check every pure-strategy profile for a profitable unilateral deviation.

Does any pure-strategy Nash equilibrium exist?

Answer: No Pure-Strategy Equilibrium

If the coins match, player 2 receives -1 and can switch to obtain 1 . If the coins differ, player 1 receives -1 and can switch to obtain 1 .

Every pure-strategy profile admits a profitable deviation. Therefore, matching pennies has **no pure-strategy Nash equilibrium**.

This conclusion concerns deterministic choices. It does not rule out an equilibrium in which players randomize.

Discussion: Battle of the Sexes

	Opera	Movie
Opera	2, 1	0, 0
Movie	0, 0	1, 2

The players prefer attending the same event to attending different events, but disagree over which event is better.

Find the pure-strategy Nash equilibria. Does the equilibrium condition select a unique event?

Answer: Two Strict Equilibria

	Opera	Movie
Opera	2, 1	0, 0
Movie	0, 0	1, 2

The two equilibria are (Opera, Opera) and (Movie, Movie). Both are strict.

Player 1 prefers the Opera equilibrium; player 2 prefers the Movie equilibrium. Mutual best responses do not determine which equilibrium the players select.

Discussion: Chicken

	H	D
H	0, 0	3, 1
D	1, 3	2, 2

H is an aggressive action and D is yielding. Find every pure-strategy Nash equilibrium.

Can a symmetric game have only asymmetric pure-strategy equilibria?

Answer: One Player Yields

	H	D
H	0, 0	3, 1
D	1, 3	2, 2

Against H , yielding gives $1 > 0$. Against D , aggression gives $3 > 2$. Thus, (H, D) and (D, H) are the two pure-strategy equilibria.

The game treats the players symmetrically, but each pure equilibrium assigns them different actions. Neither (H, H) nor (D, D) is an equilibrium.

Equilibrium and Rationalizability

In a finite two-player game, every pure-strategy Nash equilibrium uses rationalizable strategies.

1. Each equilibrium strategy is a best response to the other equilibrium strategy, so neither is strictly dominated in the original game.
2. If both survive a round, each remains a best response to a surviving opponent strategy. Neither can be strictly dominated in the reduced game.
3. Both therefore survive every round of deletion.

The converse fails: mismatched Stag Hunt profiles are rationalizable but are not equilibria.

Discussion: The Partnership Equilibrium

Two partners choose $x, y \in [0, 4]$, with $0 < c < 1/4$, and have payoffs

$$u_1 = 2(x + y + cxy) - x^2, \quad u_2 = 2(x + y + cxy) - y^2.$$

Their best responses to certain opponent efforts are

$$x = 1 + cy, \quad y = 1 + cx.$$

Solve for the Nash equilibrium. Does it coincide with the jointly optimal pair $\hat{x} = \hat{y} = 2/(1 - 2c)$?

Answer: Solve Both Best Responses

Substituting $y = 1 + cx$ into $x = 1 + cy$ gives

$$x = 1 + c(1 + cx), \quad (1 - c^2)x = 1 + c.$$

Hence, the unique Nash equilibrium is

$$x^* = y^* = \frac{1}{1 - c}.$$

Strict concavity ensures that the first-order conditions identify global best responses. This pair also equals the unique rationalizable profile. Both effort levels are below the joint optimum $2/(1 - 2c)$.

Discussion: Switching a Shared System

Two student clubs share equipment bookings. Each chooses a new booking system or the old one. These illustrative payoffs include compatibility costs.

	New	Old
New	2, 2	0, 0
Old	0, 0	1, 1

Find the pure-strategy equilibria. If both prefer the new system, can both using the old system still be an equilibrium? What happens if only one club switches?

Answer: An Inefficient Equilibrium

	New	Old
New	2, 2	0, 0
Old	0, 0	1, 1

Both (New, New) and (Old, Old) are strict equilibria. The old-system equilibrium gives both clubs less.

A club switching alone from Old to New reduces its payoff from 1 to 0. Coordinating a joint switch could benefit both; a unilateral incentive to switch is absent.

Three Strategic Tensions

- **Incentives versus joint gains.** In the prisoners' dilemma, the only equilibrium is worse for both than mutual cooperation.
- **Strategic uncertainty.** In Stag Hunt, rationalizable choices can fail to match when beliefs about the other player are wrong.
- **Inefficient coordination.** The clubs can coordinate on an inferior equilibrium, even with correct beliefs and aligned preferences.

Joint welfare refers to the players included in the model. Benefits to the players need not imply benefits to outsiders.

Self-Enforcing Agreements

Suppose the players agree to a strategy profile and each expects the others to honor it. The agreement is self-enforcing only if the profile is a Nash equilibrium.

A recommendation can coordinate which equilibrium is played. It cannot make an otherwise unprofitable action optimal when payoffs and beliefs about the others' actions stay fixed.

Equilibrium does not require an actual agreement. Nor does its definition imply that learning or repeated interaction must converge to it.

Discussion: What Would a Deviation Show?

Two participants play a one-shot prisoners' dilemma with monetary rewards. Both choose to cooperate, although defecting pays more against either action.

Does this observation prove that the participants are irrational? What assumptions would be needed for that conclusion?

How could a laboratory study distinguish a mistaken belief from a preference for helping the other participant?

Answer: Payoffs Must Represent Preferences

If monetary rewards are the players' utilities and they understand the game, cooperation contradicts best-response behavior: defection strictly dominates.

But participants may care about the other's payoff or fairness. Then the money matrix need not be their utility matrix. Confusion is another possibility; an incorrect belief alone cannot explain a strictly dominated choice.

Comprehension checks, elicited beliefs, and decisions that measure social preferences can help distinguish explanations. A laboratory controls actions and money more directly than it controls preferences or understanding.

Takeaways

Nash equilibrium requires mutual best responses.

- Check every unilateral deviation while holding the others' strategies fixed.
- Pure equilibria can be absent or multiple. Mark all best-response ties.
- Equilibrium strategies are rationalizable, but rationalizable choices need not form an equilibrium.
- Neither a unique equilibrium nor correct beliefs guarantee efficiency. The model's payoff assumptions matter when comparing predictions with behavior.