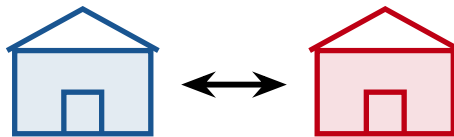


Chapter 10

Oligopoly, Tariffs, Crime, and Voting



I. Oligopoly: Cournot Quantity Competition

Two firms simultaneously choose quantities $q_1, q_2 \geq 0$ of the same product. Market price is $P = 1000 - q_1 - q_2$, and marginal cost is 100.

Each firm sells its chosen quantity and maximizes profit:

$$u_i(q_i, q_j) = (900 - q_i - q_j)q_i.$$

The stated linear price formula applies at every quantity, including negative prices at very large output. Such output will not be optimal.

A Firm's Best Response

Hold the other firm's quantity q_j fixed. Firm i maximizes

$$(900 - q_j)q_i - q_i^2.$$

The first derivative is $900 - q_j - 2q_i$; the second is -2 . Including the nonnegativity constraint gives

$$\text{BR}_i(q_j) = \max\{0, (900 - q_j)/2\}.$$

A higher rival output lowers optimal own output. Quantities are **strategic substitutes**.

The Cournot Equilibrium

An equilibrium cannot have $q_i = 0$: its rival would choose 450, to which zero is not a best response. Both equilibrium quantities are positive.

$$2q_1 + q_2 = 900, \quad q_1 + 2q_2 = 900.$$

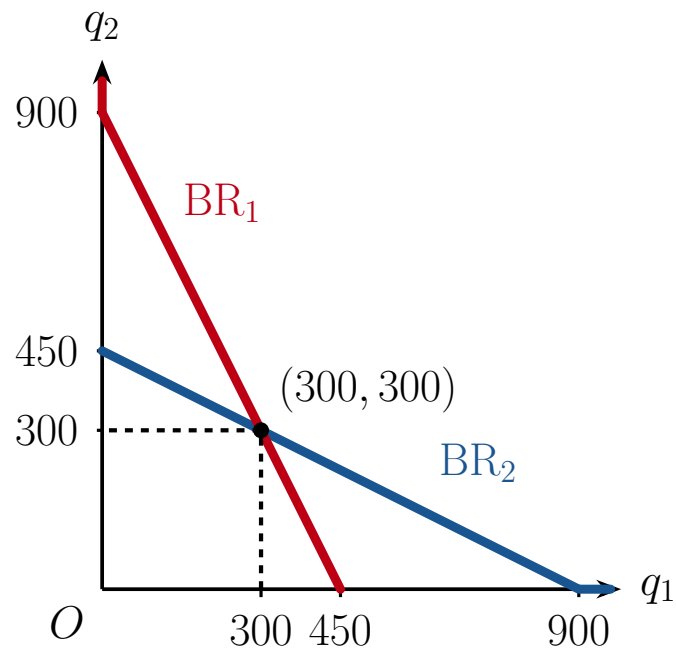
Subtracting gives $q_1 = q_2$, hence $q_1^* = q_2^* = 300$.

Total output is 600, price is 400, and each firm earns

$$(400 - 100) \cdot 300 = 90,000.$$

Best Responses in Quantity Space

The intersection makes both firms best respond at once.



A Joint Output Agreement

Suppose the Cournot firms agree to produce 225 each. At $(225, 225)$, price is 550 and each firm earns $(550 - 100) \cdot 225 = 101,250 > 90,000$.

But a best response to 225 is $(900 - 225)/2 = 337.5$, not 225. The agreement is not self-enforcing.

Writing $Q = q_1 + q_2$, joint profit is $(900 - Q)Q$, maximized at $Q = 450$. Each firm ignores the loss its extra output imposes on its rival.

Different Production Costs

Suppose $P = 12 - q_1 - q_2$, with marginal costs 2 for firm 1 and 4 for firm 2. Their best responses are

$$\text{BR}_1(q_2) = \max\{0, (10 - q_2)/2\}, \quad \text{BR}_2(q_1) = \max\{0, (8 - q_1)/2\}.$$

Solving $2q_1 + q_2 = 10$ and $q_1 + 2q_2 = 8$ gives

$$(q_1^*, q_2^*) = (4, 2), \quad P^* = 6.$$

Both are positive and satisfy the best-response formulas. The firms earn 16 and 4; the lower-cost firm serves two-thirds of the market.

Bertrand Price Competition

Two firms now simultaneously choose prices $p_1, p_2 \geq 0$. The good is identical, and each firm has marginal cost 100 and no capacity limit.

Demand at the lower price p is $Q(p) = \max\{1000 - p, 0\}$. The lower-priced firm supplies all demand; equal prices split it equally.

Firm i earns $(p_i - 100)Q(p_i)$ if it is cheaper, half that amount if prices tie, and zero if it is more expensive.

Undercutting a Rival

At $(400, 400)$, each earns $300 \cdot 600/2 = 90,000$. Undercutting to 399 gives profit $299 \cdot 601 = 179,699$.

At $(100, 100)$, both earn zero. Raising price loses all demand and still gives zero; lowering price generates sales at a loss. So $(100, 100)$ is an equilibrium.

A tiny price cut can capture the entire market. A tiny quantity increase only adds a tiny amount to sales.

Why No Other Price Pair Is an Equilibrium

If the lowest price is below 100, a firm making sales loses money and can gain by switching to 100.

If the lowest price is above 100, a firm can profitably undercut it when sales are positive; if no one buys, a price of 550 earns positive profit.

If the lowest price is 100 and the other price is higher, the cheaper firm can raise its price slightly and earn positive profit while staying cheaper. Thus, the unique equilibrium is $(100, 100)$.

An Equilibrium Without Best Responses Everywhere

Against a rival price of 200, prices just below 200 give profits approaching $800 \cdot 100 = 80,000$.

No such price attains 80,000: profit rises as the price approaches 200 from below. At exactly 200, demand is split and profit drops to 40,000.

So a best response to 200 does not exist. Equilibrium only requires best responses at the prices actually chosen in equilibrium.

Discussion: What Does the Comparison Hold Fixed?

Cournot gives price 400 and profit 90,000 per firm; Bertrand gives price 100 and zero profit in the same demand and cost example.

Which assumptions let a small undercut capture all customers? Would that argument survive a capacity limit or different products?

Is an outcome maximizing the firms' joint profit necessarily best for consumers?

Answer: The Choice Variable Matters

The Bertrand argument uses identical goods, buyers choosing the cheaper seller, and enough capacity to serve all demand.

Capacity limits leave some demand for a higher-priced rival. Product differences can make buyers stay with a more expensive firm.

Joint profit counts firms' payoffs. It omits consumer surplus, so a joint profit maximum is not a claim about total social welfare.

II. Tariffs: Tariff Setting

Two countries simultaneously choose tariffs $x_i \in [0, 100]$. In this illustrative model, country i 's payoff is

$$u_i(x_i, x_j) = 2000 + 60x_i + x_i x_j - x_i^2 - 90x_j.$$

An own tariff can benefit the country, while a rival's tariff reduces its payoff at the relevant outcomes.

The cross partial derivative is $1 > 0$. Tariffs are strategic complements: a higher rival tariff raises the marginal incentive to impose one's own tariff.

Equilibrium Tariffs and Mutual Restraint

The first-order condition is $60 + x_j - 2x_i = 0$; the second derivative is -2 . Thus, $BR_i(x_j) = 30 + x_j/2 \in [30, 80]$ and the unique equilibrium is $(60, 60)$.

The symmetric payoff comparisons are shown below.

Each tariff	60	30	0
Each payoff	200	1100	2000

Against zero, a best response is 30. Mutual restraint benefits both but requires more than a nonbinding promise.

III. Crime: Crime and Enforcement

The government chooses enforcement $x > 0$; a criminal chooses activity $y \geq 0$. Let $c > 0$ measure enforcement cost. Their payoffs are

$$u_G(x, y) = -xc^4 - \frac{y^2}{x}, \quad u_C(x, y) = \frac{\sqrt{y}}{1 + xy}.$$

Enforcement is costly and reduces the harm from crime. Here $\sqrt{y}$ is the value of criminal activity when the criminal is not caught, and $1/(1 + xy)$ is the probability that the criminal evades capture: higher enforcement x or activity y raises the chance of capture.

Positive enforcement is imposed here because the government's formula is undefined at $x = 0$.

Best Responses in Crime and Enforcement

For $y > 0$, the government has first derivative $-c^4 + y^2/x^2$ and second derivative $-2y^2/x^3 < 0$. Its unique best response is $x = y/c^2$.

For $x > 0$ and $y > 0$, the criminal's derivative is

$$\frac{1 - xy}{2\sqrt{y}(1 + xy)^2}.$$

It is positive below $y = 1/x$ and negative above. Since positive activity beats zero, the criminal's unique best response is $y = 1/x$.

Equilibrium Crime and Enforcement

Substituting $y = 1/x$ into $x = y/c^2$ gives $x^2 = 1/c^2$. The unique equilibrium is

$$x^* = 1/c, \quad y^* = c.$$

Higher enforcement cost reduces enforcement and increases crime. Since $x^*y^* = 1$, the probability of evading capture stays at $1/2$.

Zero crime cannot be an equilibrium: at every $x > 0$, the criminal gains by choosing positive activity. The model does not predict complete deterrence.

IV. Voting: Candidates Choose Policy Positions

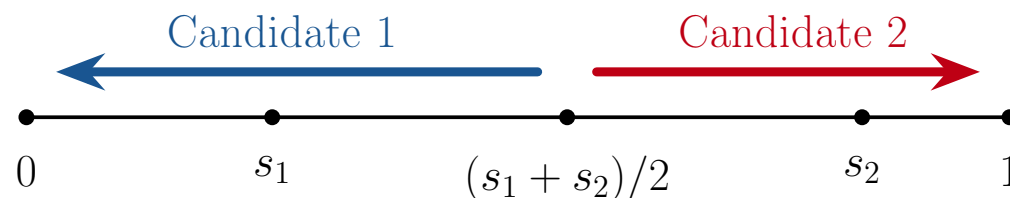
Two candidates simultaneously choose positions $s_1, s_2 \in [0, 1]$. Voters have ideal points with continuous, strictly increasing cumulative distribution F , where $F(0) = 0$ and $F(1) = 1$.

Every voter votes for the closer candidate; ties split equally. A candidate's payoff is 1 for winning, $1/2$ for tying, and 0 for losing.

The unique median m satisfies $F(m) = 1/2$. Strict increase rules out empty intervals of voter locations.

The Voter Who Separates the Candidates

For $s_1 < s_2$, voters left of $(s_1 + s_2)/2$ choose candidate 1. Her vote share is $F((s_1 + s_2)/2)$.



The relevant midpoint separates voters between candidates. It need not coincide with the median of the voter distribution.

Why Both Candidates Locate at the Median

If both choose m , any deviation puts the midpoint strictly on the deviator's side of m . The deviator receives less than half the votes.

If both choose $s \neq m$, either can move slightly toward m and win. If their positions differ and one loses, the loser can match the winner and obtain a tie instead.

If distinct positions tie, moving slightly toward the rival produces a majority. Therefore, the unique equilibrium is (m, m) .

A Nonuniform Electorate

Suppose $F(t) = t^2$ for $t \in [0, 1]$. The median solves $m^2 = 1/2$, so $m = 1/\sqrt{2} \approx 0.707$. Both candidates choose this position in equilibrium.

If both are at $1/2$, one can move to $3/4$. Their midpoint is $5/8$. The candidate on the right receives vote share

$$1 - F(5/8) = 1 - 25/64 = 39/64 > 1/2.$$

The median depends on where voters are located, not just on the endpoints of the policy space.

What Supports Median-Voter Convergence?

The two-candidate model fixes who votes and makes candidates care only about winning. Voters know the platforms and choose the closer one.

- A candidate who also values the implemented policy may accept a lower chance of winning to advocate a preferred position.
- A primary can reward positions favored by a different electorate from the one voting in the general election.
- If platforms affect turnout or campaign support, moving toward the median can lose supporters as well as attract voters.

These changes alter incentives. Uncertainty about platforms or voter preferences also changes the calculations behind convergence.

Example: Three Candidates

Three candidates choose positions in $[0, 1]$. Voters are uniform and vote for the closest candidate, splitting equally among tied candidates. Here each candidate maximizes her *vote share*, rather than a win/tie/loss payoff.

If positions are distinct, write them as $a < b < c$. The left candidate's share is $(a + b)/2$. Moving from a toward b , while staying left of it, increases her share. So distinct positions cannot form an equilibrium.

If exactly two candidates share a position, the lone candidate can move toward them while remaining separate. Her vote share increases for the same reason.

Sharing a Platform Does Not Resolve the Problem

If all three candidates choose s , each receives $1/3$ of the votes. A candidate moving just to the left of s receives nearly s ; moving just to the right receives nearly $1 - s$.

At least one of s and $1 - s$ is at least $1/2$. A sufficiently small move toward that side therefore raises her share above $1/3$. At $s = 0$ she can move right; at $s = 1$ she can move left.

All three possible position patterns admit a profitable deviation. Thus, this three-candidate vote-share game has no pure-strategy Nash equilibrium. The two-candidate convergence result does not extend to it.

Example: Voting or Abstaining

Two candidates are already on the ballot. A citizen prefers A to B and chooses whether to vote for A or abstain. Voting costs $k > 0$. A tied election is decided by a fair coin.

Let $V > 0$ be her utility gain if A wins rather than B . Her vote changes the winning probability only when the other votes leave A tied or exactly one vote behind.

Let r be the probability of these two events together, given her beliefs about the other voters. Her expected net gain from voting is

$$\frac{rV}{2} - k.$$

Voting is optimal when this is nonnegative; abstaining is optimal when it is nonpositive.

Pivotality and Participation

Suppose $V = 100$, $k = 1$, and the citizen assigns probability $r = 1/1000$ to being pivotal. Voting changes her expected outcome utility by only

$$\frac{rV}{2} = 0.05 < 1.$$

She abstains even though she strongly prefers A to win.

If voting itself gives a benefit b , such as satisfaction from civic participation, the net gain becomes $rV/2 + b - k$. A sufficiently large b can make voting optimal even when pivotality is unlikely.

Turnout depends on preferences over participation as well as outcomes. The probability r comes from beliefs about others' voting behavior; it is not fixed by election size alone.

Strategic Voting

Three voting blocs L, M, C contain 40%, 30%, and 30% of voters, respectively. Each bloc votes together for one candidate: A, B , or C . The candidate with the most votes wins.

The preference rankings in this illustrative election are

Bloc	First	Second	Third
L	A	B	C
M	B	A	C
C	C	B	A

Payoffs

Each bloc's payoff adds a voting benefit and an outcome benefit. The voting benefit is 2, 1, or 0 for voting for its first, second, or third choice, regardless of who wins.

The outcome benefit is 4, 2, or 0 if its first, second, or third choice wins, regardless of its own vote. A bloc can therefore like its own vote but dislike the winner.

Voting for a Second Choice

Write votes in the order (L, M, C) . At (A, B, C) , all blocs vote for their first choice. The votes are split, so the larger bloc L elects A . Bloc C ranks $C > B > A$: its own vote gives 2, but the winner gives 0. Its total payoff is $2 + 0 = 2$.

Hold the other votes fixed. If bloc C votes for B , the profile becomes (A, B, B) . Blocs M and C now elect B . Bloc C gets $1 + 2 = 3$: it loses 1 in voting benefit but gains 2 from a better winner.

Checking the Strategic-Voting Equilibrium

At (A, B, B) , check that no bloc wants to change its vote. Bloc M receives its maximum payoff, $2 + 4 = 6$. Bloc L cannot change the winner, so voting for A gives its highest voting benefit and total payoff $2 + 2 = 4$.

Bloc C receives 3; switching to A gives 0 and switching to C gives 2. Thus, (A, B, B) is an equilibrium. Sincere voting need not be optimal.

Example: Congestion at Two Locations

Ten firms simultaneously choose Downtown or Suburbs. If n choose Downtown, each earns $5n - n^2 + 50$. If m choose Suburbs, each earns $48 - m$, with $n + m = 10$.

A move changes both location counts. A Downtown firm moving to Suburbs would face $m + 1 = 11 - n$ firms there; a Suburbs firm moving Downtown would face $n + 1$ firms.

Equilibrium compares current profit with profit after such a move.

Compare Payoffs After a Move

For $0 < n < 10$, no Downtown firm wants to move if

$$5n - n^2 + 50 \geq 48 - (11 - n) = 37 + n.$$

No Suburbs firm wants to move if

$$38 + n \geq 5(n + 1) - (n + 1)^2 + 50 = 54 + 3n - n^2.$$

For integer n , these require $n \leq 6$ and $n \geq 6$. The endpoints 0 and 10 admit profitable moves. Thus, $n = 6$, $m = 4$; each earns 44. Equilibrium need not equalize payoffs; it compares payoffs after a move.

Takeaways

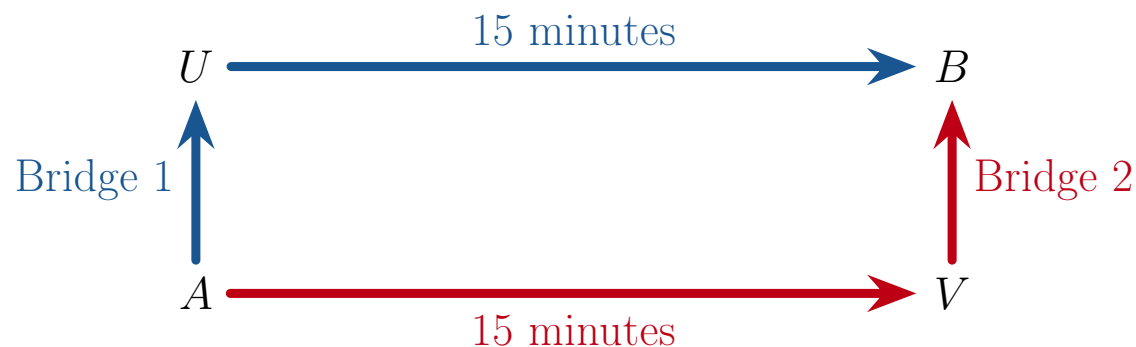
Equilibrium depends on choices, constraints, and external effects.

- Quantity and price competition create different deviation incentives.
- An interior first-order condition needs boundary and payoff checks.
- Joint gains do not make an agreement self-enforcing. A new road can worsen equilibrium congestion even while expanding feasible choices.
- Candidate convergence depends on objectives and the electorate. Voting incentives depend on pivotality and the value of participation.

Additional Example: The Transportation Paradox

There are 1,000 drivers traveling from A to B . Each simultaneously chooses a route and minimizes travel time. A bridge carrying F cars takes $F/100$ minutes; each ordinary road takes 15 minutes.

The two routes each combine one bridge with one ordinary road:



Route 1 is $A \rightarrow U \rightarrow B$; Route 2 is $A \rightarrow V \rightarrow B$.

Equilibrium Before the New Road

Let R_1 and R_2 be the numbers of drivers on the two routes, including the driver whose travel time is being calculated. With $R_1 + R_2 = 1000$,

$$T_1 = 15 + \frac{R_1}{100}, \quad T_2 = 15 + \frac{R_2}{100}.$$

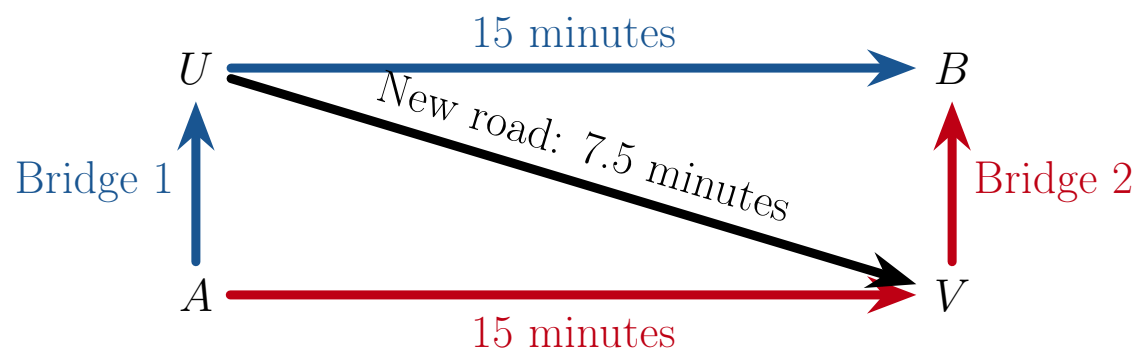
At $R_1 = R_2 = 500$, everyone takes 20 minutes. A driver switching routes would join 501 drivers and take 20.01 minutes, so no one gains.

Any unequal split has at least 501 drivers on the busier route and at most 499 on the other. A driver on the busier route can reduce travel time by switching.

Thus, the unique pure-equilibrium route counts are (500, 500). Many strategy profiles produce these counts.

A Faster Road Creates a Third Route

Add a one-way road from U to V taking 7.5 minutes. Drivers can now choose Route 3, $A \rightarrow U \rightarrow V \rightarrow B$, which uses both bridges. The original roads remain available.



Starting from the old split, one driver switching to Route 3 takes $5 + 7.5 + 5.01 = 17.51$ minutes. The old profile is no longer an equilibrium.

More Drivers on Both Bridges

With route counts $R_1 + R_2 + R_3 = 1000$, Bridge 1 carries $R_1 + R_3$ cars and Bridge 2 carries $R_2 + R_3$. Travel times are

$$T_1 = 15 + (R_1 + R_3)/100,$$

$$T_2 = 15 + (R_2 + R_3)/100,$$

$$T_3 = 7.5 + (R_1 + R_3)/100 + (R_2 + R_3)/100.$$

Seek a symmetric candidate with $R_1 = R_2 = r$ and $R_3 = 1000 - 2r$. Equal travel times require

$$15 + \frac{1000 - r}{100} = 7.5 + 2\frac{1000 - r}{100}.$$

This gives $(R_1, R_2, R_3) = (250, 250, 500)$. Both bridges carry 750 cars, and all three routes take 22.5 minutes.

Checking Each Driver's Incentive

At (250, 250, 500), a driver's alternative times include the change in bridge traffic caused by her own move:

Current route	Alternative route	Time after moving
1	2 or 3	22.51
2	1 or 3	22.51
3	1 or 2	22.50

No driver can improve on 22.5 minutes, so this is a Nash equilibrium. Everyone takes longer than the original 20 minutes.

Drivers ignore the delay they add to other bridge users. A new road can therefore change equilibrium behavior and make everyone worse off: the **Braess paradox**.

A Faster Link Can Worsen Congestion

If the new link takes 5 minutes, all 1,000 drivers using Route 3 is an equilibrium. Each takes $10 + 5 + 10 = 25$ minutes. Switching to Route 1 or 2 still takes $10 + 15 = 25$ minutes, so no driver gains.

The equilibrium examples give the following comparison:

Network	Route counts (R_1, R_2, R_3)	Minutes
Original	(500, 500, 0)	20
7.5-minute link	(250, 250, 500)	22.5
5-minute link	(0, 0, 1000)	25

The old 20-minute allocation remains feasible without using the link. The best travel times a planner could achieve cannot worsen.