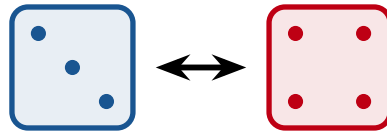


Chapter 11

Mixed-Strategy Nash Equilibrium



Example: Matching Pennies

Two players simultaneously choose H or T . Player 1 wins when choices match; player 2 wins when they differ. The payoff matrix is

	H	T
H	$1, -1$	$-1, 1$
T	$-1, 1$	$1, -1$

At every pure profile, the loser gains by switching. So there is no pure-strategy equilibrium.

Randomization Can Be a Best Response

Suppose player 2 chooses H and T with equal probabilities. Player 1's expected payoffs are

$$u_1(H) = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot (-1) = 0, \quad u_1(T) = \frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 1 = 0.$$

The same calculation gives player 2 expected payoff zero from either action when player 1 mixes equally. Thus, both mixing equally is an equilibrium.

Either player mixing equally guarantees expected payoff zero against any opponent mixture. Randomization prevents the opponent from exploiting a predictable choice.

Mixed-Strategy Nash Equilibrium

Definition. A mixed-strategy profile $\sigma = (\sigma_1, \dots, \sigma_n)$ is a **Nash equilibrium** if each σ_i maximizes player i 's expected payoff given σ_{-i} .

Equivalently, for every player i and pure strategy s_i ,

$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(s_i, \sigma_{-i}).$$

Players randomize independently. Pure strategies are included as degenerate mixed strategies that put probability 1 on a single action.

Support and Indifference

Definition. The **support** of a mixed strategy is the set of pure strategies assigned positive probability.

A mixture is a best response exactly when every action in its support is a pure best response. Expected payoff is a probability-weighted average.

Thus, used actions must tie for the highest payoff. An unused action may tie that payoff, but it must not give more.

Whose Probability Makes Whom Indifferent?

Let p be player 1's probability of H and q player 2's. In matching pennies,

$$u_1(H) = 2q - 1, \quad u_1(T) = 1 - 2q.$$

Player 1 is willing to mix only when $q = 1/2$. Similarly, player 2 is willing to mix only when $p = 1/2$.

A player's indifference equation determines the *opponent's* probability. The player's own randomization does not equalize her pure-action payoffs.

Best Responses as Mixing Probabilities

Here $p = \Pr_1(H)$ and $q = \Pr_2(H)$; both are constrained to $[0, 1]$.

$$\text{BR}_1(q) = \begin{cases} \{0\}, & q < 1/2, \\ [0, 1], & q = 1/2, \\ \{1\}, & q > 1/2. \end{cases} \quad \text{BR}_2(p) = \begin{cases} \{1\}, & p < 1/2, \\ [0, 1], & p = 1/2, \\ \{0\}, & p > 1/2. \end{cases}$$

The only mutual best response is $(p, q) = (1/2, 1/2)$. Against a pure strategy the opponent has a unique pure best response, so no equilibrium has just one player mixing here.

Example: Lobbying for a Favorable Decision

Two firms choose Lobby (L) or Not (N). Lobbying costs 15. Both lobbying or neither lobbying yields a neutral decision worth 10 each. Lobbying alone gives firm 1 a gross benefit of 40 or firm 2 a gross benefit of 30; the other firm gets zero. The resulting net payoffs are

	L	N
L	$-5, -5$	$25, 0$
N	$0, 15$	$10, 10$

Asymmetric Mixing Probabilities

The pure equilibria are (L, N) and (N, L) . Let p be firm 1's probability of lobbying and q firm 2's. If both mix, indifference requires

$$q \cdot (-5) + (1 - q) \cdot 25 = (1 - q) \cdot 10,$$

$$p \cdot (-5) + (1 - p) \cdot 15 = (1 - p) \cdot 10.$$

Thus, $q = 3/4$ and $p = 1/2$. The expected payoffs are $(1 - q)10 = 5/2$ for firm 1 and $(1 - p)10 = 5$ for firm 2.

Every pure opponent action has a unique best response, so these three profiles exhaust the equilibria.

Discussion: Changing a Mixing Probability

At the lobbying equilibrium $(p, q) = (1/2, 3/4)$, suppose firm 1 changes its probability of lobbying to $p = 1$, while firm 2 keeps $q = 3/4$.

Does firm 1 gain? Is the new profile also an equilibrium?

Explain how a player can be indifferent among all her mixtures even though only one mixture is part of an equilibrium with the other player's mix.

Answer: A Best Response Is Not a Whole Equilibrium

Against $q = 3/4$, either pure action gives firm 1 payoff $5/2$. Every mixture gives the same payoff, so changing p does not help it.

But against $p = 1$, firm 2 strictly prefers N , which gives zero rather than -5 . Its old mixture $q = 3/4$ is no longer a best response.

A unilateral change can preserve the deviator's optimality while destroying the other player's optimality. Equilibrium requires both.

Example: Entry with Different Fixed Costs

Two firms simultaneously choose Enter (E) or Stay Out (O). Firm i pays a fixed cost $c_i \in (0, 1)$ only if it enters. Revenue is 1 for a lone entrant and 0 for each firm if both enter.

The net payoff matrix, with firm 1 choosing the row, is

	E	O
E	$-c_1, -c_2$	$1 - c_1, 0$
O	$0, 1 - c_2$	$0, 0$

The pure equilibria are (E, O) and (O, E) : either firm can be the sole entrant. Entering against an entrant loses money; entering alone is profitable.

Entry Probabilities and Indifference

Let p_i be firm i 's probability of entry. Firm 1's expected payoff from entering is

$$p_2(-c_1) + (1 - p_2)(1 - c_1) = 1 - c_1 - p_2.$$

Staying out gives zero. For firm 1 to mix, $p_2 = 1 - c_1$. Applying the same reasoning to firm 2 gives

$$p_1^* = 1 - c_2, \quad p_2^* = 1 - c_1.$$

Both probabilities lie strictly between zero and one. Each action yields zero expected payoff, so both firms are best responding.

The Higher-Cost Firm Enters More Often

Suppose $c_1 = 1/4$ and $c_2 = 3/4$. The mixed equilibrium has

$$p_1^* = 1/4, \quad p_2^* = 3/4.$$

Firm 2 has the higher cost but enters more often. Its frequent entry makes firm 1 just indifferent despite firm 1's low cost. Firm 1's rare entry gives firm 2 enough chance of being alone to cover its high cost.

Holding the rival's entry probability fixed, a higher own cost still makes entry less attractive. The equilibrium comparison instead adjusts the rival's probability to preserve indifference.

This conclusion concerns the mixed equilibrium. Both pure equilibria also remain possible; the model does not select among the three.

Example: A Tennis-Service Game

The server (player 1) and receiver (player 2) choose Forehand (F), Center (C), or Backhand (B): serve direction and defensive position. These illustrative payoffs favor different combinations.

	F	C	B
F	0, 5	2, 3	2, 3
C	2, 3	0, 5	3, 2
B	5, 0	3, 2	2, 3

Dominance Before Indifference

For the server, mixing C with probability $1/4$ and B with probability $3/4$ gives payoffs against F, C, B , respectively,

$$\frac{1}{4} \cdot 2 + \frac{3}{4} \cdot 5 = 17/4, \quad \frac{1}{4} \cdot 0 + \frac{3}{4} \cdot 3 = 9/4, \quad \frac{1}{4} \cdot 3 + \frac{3}{4} \cdot 2 = 9/4.$$

These strictly exceed F 's payoffs $(0, 2, 2)$, so remove the server's F .

Against the remaining rows C, B , the receiver gets $(5, 2)$ from C and $(3, 0)$ from F . Thus, remove the receiver's F as well.

The Reduced Tennis Game

The surviving payoff matrix is

	C	B
C	0, 5	3, 2
B	3, 2	2, 3

Let p be the server's probability of C and q the receiver's. Indifference gives

$$\begin{aligned}(1 - q) \cdot 3 &= q \cdot 3 + (1 - q) \cdot 2, \\ p \cdot 5 + (1 - p) \cdot 2 &= p \cdot 2 + (1 - p) \cdot 3.\end{aligned}$$

So $p = q = 1/4$. There are no pure equilibria in this reduced game.

Checking the Excluded Actions

At the tennis equilibrium, the server's used actions give payoff $9/4$. Her excluded F gives

$$\frac{1}{4} \cdot 2 + \frac{3}{4} \cdot 2 = 2 < 9/4.$$

The receiver's used actions give payoff $11/4$. His excluded F gives

$$\frac{1}{4} \cdot 3 + \frac{3}{4} \cdot 0 = 3/4 < 11/4.$$

Thus, both $(0, 1/4, 3/4)$ are best responses in the original game.

Example: Protecting Three Targets

An attacker and a defender simultaneously choose one of three targets, with values 2, 3, and 4. A protected attack gives both zero; otherwise, the attacker gets the target's value and the defender its negative.

Rows are attack choices and columns are defense choices:

	D_2	D_3	D_4
A_2	0, 0	2, -2	2, -2
A_3	3, -3	0, 0	3, -3
A_4	4, -4	4, -4	0, 0

A pure profile cannot be an equilibrium: an undefended attack invites a change in defense; a defended attack invites a different target.

Making Three Attack Choices Indifferent

Let q_v be the probability of defending the target with value v . To construct an equilibrium using all three targets, the attacker's payoffs must satisfy

$$2(1 - q_2) = 3(1 - q_3) = 4(1 - q_4) = U.$$

Thus, $q_v = 1 - U/v$. The probabilities sum to one, so

$$3 - U \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} \right) = 1, \quad U = \frac{24}{13}.$$

The defense probabilities are

$$(q_2, q_3, q_4) = \left(\frac{1}{13}, \frac{5}{13}, \frac{7}{13} \right).$$

The most valuable target receives the most protection.

Making Three Defense Choices Indifferent

Let p_v be the probability of attacking target v . Protecting it avoids expected damage vp_v . All three defense choices are optimal when

$$2p_2 = 3p_3 = 4p_4, \quad p_2 + p_3 + p_4 = 1.$$

So the attack probabilities are

$$(p_2, p_3, p_4) = \left(\frac{6}{13}, \frac{4}{13}, \frac{3}{13} \right).$$

Against these probabilities, each defense gives payoff $-24/13$. Against q , each attack gives $24/13$. All probabilities are positive and sum to one; all available actions are best responses.

These mixtures form an equilibrium. The most valuable target is attacked least often: fewer attacks suffice to make its protection worthwhile.

A Procedure for Finding All Equilibria

1. Find pure equilibria and eliminate strictly dominated actions.
2. Select candidate supports. Include cases where one player chooses a single action with probability 1 while the other randomizes.
3. Equate payoffs for each player's used actions and solve for the other player's probabilities.
4. Check nonnegative probabilities summing to one, and verify that no unused action gives a higher payoff.

To claim *all* equilibria, cover every possible support or explain why omitted supports cannot work.

Discussion: An Indifference Trap

Consider this zero-sum game, with player 1's payoff first:

	L	R
U	$2, -2$	$0, 0$
D	$0, 0$	$2, -2$
S	$2, -2$	$2, -2$

A proposed equilibrium has player 1 mix equally over U, D and player 2 mix equally over L, R . The used actions are indifferent. Is it an equilibrium?

Answer: Check Every Unused Action

Against equal probabilities on L, R , player 1 gets

$$u_1(U) = 1, \quad u_1(D) = 1, \quad u_1(S) = 2.$$

Switching to S raises her payoff from 1 to 2. So the proposed profile is not an equilibrium.

Equating the payoffs of the proposed support only establishes indifference *within* that support. It does not establish optimality.

Example: One Player Mixes

Consider the game below. Let $p = \Pr_1(A)$ and $q = \Pr_2(L)$.

	L	R
A	2, 2	0, 0
B	2, 0	1, 1

Against L , both A and B give player 1 payoff 2, so any mixture is a best response. To form an equilibrium, her mixture must also make L optimal for player 2.

A Continuum with One Player Pure

Player 1's payoffs are $2q$ from A and $1 + q$ from B . She can mix only if $q = 1$; then every $p \in [0, 1]$ is optimal.

Player 2 gets $2p$ from L and $1 - p$ from R . So L is a best response when $p \geq 1/3$.

All equilibria are $q = 1$ with $p \in [1/3, 1]$, together with $(p, q) = (0, 0)$. Indeed, if $q < 1$, player 1 must choose B , which uniquely induces R . At $p = 1/3$, player 2 is indifferent but can still choose L purely.

Existence in Finite Games

Theorem. *Every game with finitely many players and finitely many pure strategies has at least one Nash equilibrium in pure or mixed strategies.*

This does not imply a pure equilibrium, a unique equilibrium, or that every rationalizable strategy appears in equilibrium.

The theorem's finite-strategy assumption matters. A game with continuous prices requires its own existence argument.

Example: No Best Response

Each of two players simultaneously chooses a positive integer n_i . Player i receives $u_i(n_i, n_j) = 1 - 1/n_i$, regardless of the other's choice. Payoffs are bounded above by 1.

Any pure choice can be improved by choosing the next integer:

$$u_i(n_i + 1, n_j) - u_i(n_i, n_j) = \frac{1}{n_i(n_i + 1)} > 0.$$

Randomization cannot help. For any distribution over integers, shifting each chosen integer from n to $n + 1$ raises payoff at every realization and hence raises expected payoff strictly.

There is no best response, even among mixed strategies, and thus no Nash equilibrium. A payoff supremum need not be attained when the action set is infinite.

Advanced Example: Price Competition with Capacity Limits

Two firms simultaneously set prices $p_i \geq 0$ and have zero production cost. Ten customers each buy one unit at a price at most 1. Each firm can supply at most eight units.

If $p_i < p_j \leq 1$, firm i serves eight and firm j serves two. If $p_i = p_j \leq 1$, each serves five. A price above 1 makes no sales.

Charging 1 guarantees at least two sales and payoff at least 2. So prices above 1 cannot be best responses.

No Pure Price Pair Survives

At a common positive price p , each sells five. A sufficiently small undercut sells eight and gives $8(p - \varepsilon) > 5p$.

At $(0, 0)$, raising price to 1 gives payoff 2 instead of zero.

If $p_i < p_j \leq 1$, the lower-priced firm can raise its price while remaining below p_j , keeping eight customers and increasing profit. Thus, there is no pure-strategy equilibrium.

Constructing a Continuous Mixed Strategy

Seek a symmetric equilibrium with a continuous price distribution F on an interval $[a, 1]$. Here $F(p)$ is the probability the rival charges at most p ; ties have probability zero.

At price $p \in [a, 1]$, expected profit is

$$(1 - F(p)) \cdot 8p + F(p) \cdot 2p.$$

At $p = 1$, the firm earns 2. Indifference throughout the support therefore requires $8p - 6pF(p) = 2$.

The Equilibrium Price Distribution

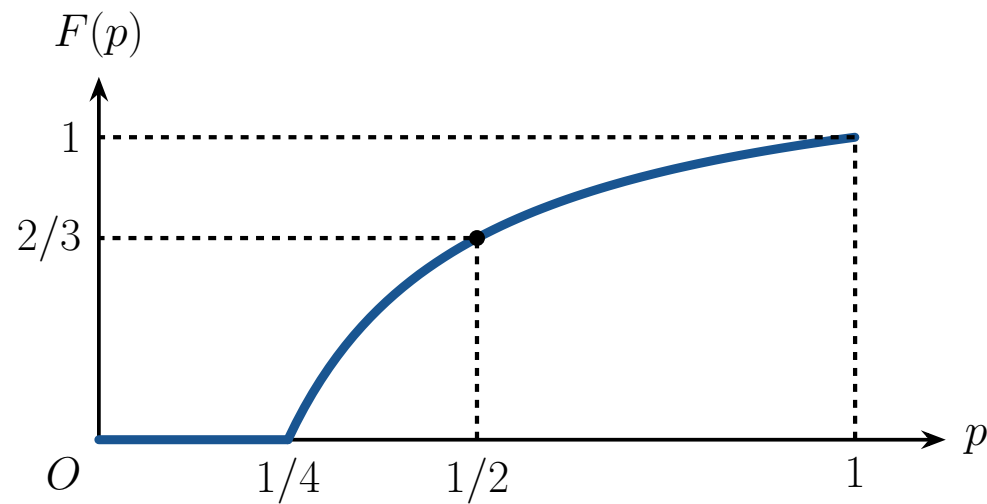
Solving the indifference equation gives $F(p) = (4p - 1)/(3p)$. The lower endpoint satisfies $F(a) = 0$, hence $a = 1/4$.

$$F(p) = \begin{cases} 0, & p < 1/4, \\ \frac{4p - 1}{3p}, & 1/4 \leq p \leq 1, \\ 1, & p > 1. \end{cases}$$

This is a valid continuous distribution: it rises from 0 to 1, with density $f(p) = 1/(3p^2) > 0$ on $(1/4, 1)$.

Reading the Price Distribution

The cumulative distribution gives probabilities of price ranges.



Each firm charges at most $1/2$ with probability $2/3$.

Verifying All Price Deviations

For $0 \leq p < 1/4$, the firm is always cheaper, selling eight units and earning $8p < 2$.

For $1/4 \leq p \leq 1$, substitution gives expected profit $8p - 6pF(p) = 2$. For $p > 1$, it makes no sales and earns zero.

Every pure deviation gives at most 2, so any probability-weighted average of those payoffs is also at most 2. Thus, both using F is a Nash equilibrium. This verifies existence without claiming uniqueness.

Takeaways

Mixing requires indifference and optimality at the same time.

- Every action used with positive probability must be a best response.
- The opponent's probabilities determine one's indifference; equilibrium entry and attack frequencies can therefore be counterintuitive.
- Check unused actions, feasibility, and boundary supports.
- Finite games always have an equilibrium in pure or mixed strategies; infinite-action games require separate verification.